

TWO INTERESTING INEQUALITIES FOR ACUTE TRIANGLES

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Abstract. The paper considers the proofs of two interesting geometric inequalities for acute triangles.

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In this paper we will prove two interesting inequalities for acute triangles, namely:

$$\frac{AH}{a} + \frac{BH}{b} + \frac{CH}{c} \geq \sqrt{3} \quad (1)$$

and

$$\frac{AH}{a} \cdot \frac{BH}{b} \cdot \frac{CH}{c} \leq \frac{1}{3\sqrt{3}}, \quad (2)$$

where the point H is the orthocentre of the $\triangle ABC$ with sides a , b and c . For the purpose we will make use of the following two inequalities:

$$\sin \alpha \sin \beta \sin \gamma \leq \frac{3\sqrt{3}}{8} \quad (3)$$

and

$$a^2 + b^2 + c^2 \geq 4F\sqrt{3}, \quad (4)$$

where α , β and γ are the interior angles of the $\triangle ABC$ and F is the area of the triangle.

Remark 1. The proofs of the inequalities (3) and (4) can be found in (Bottema et al., 1969), (Bulajich Manfrino et al., 2009), (Grozdev, 2007) and (Cvetkovski, 2012). We will propose new proofs of the which are not included in the above references.

We start with the inequality (3).

Firstly, we will prove the inequality:

$$\sin x \sin y \leq \sin^2 \frac{x+y}{2}; \quad (x, y \in [0, \pi]). \quad (5)$$

We have

$$\sin x \sin y \leq \sin^2 \frac{x+y}{2}$$

$$\begin{aligned}
 &\Leftrightarrow 2 \sin \frac{x}{2} \cos \frac{x}{2} \cdot 2 \sin \frac{y}{2} \cos \frac{y}{2} \leq \left(\sin \frac{x}{2} \cos \frac{y}{2} + \cos \frac{x}{2} \sin \frac{y}{2} \right)^2 \\
 &\Leftrightarrow 4 \sin \frac{x}{2} \cos \frac{x}{2} \sin \frac{y}{2} \cos \frac{y}{2} \leq \sin^2 \frac{x}{2} \cos^2 \frac{y}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2} \sin \frac{y}{2} \cos \frac{y}{2} + \cos^2 \frac{x}{2} \sin^2 \frac{y}{2} \\
 &\Leftrightarrow \sin^2 \frac{x}{2} \cos^2 \frac{y}{2} - 2 \sin \frac{x}{2} \cos \frac{y}{2} \cos \frac{x}{2} \sin \frac{y}{2} + \cos^2 \frac{x}{2} \sin^2 \frac{y}{2} \geq 0 \\
 &\Leftrightarrow \left(\sin \frac{x}{2} \cos \frac{y}{2} - \cos \frac{x}{2} \sin \frac{y}{2} \right)^2 \geq 0 \\
 &\Leftrightarrow \sin^2 \left(\frac{x}{2} - \frac{y}{2} \right) \geq 0.
 \end{aligned}$$

The last inequality is obvious, which proves the inequality (5). Note, that equality holds if and only if $x = y$.

The next two inequalities follow from the inequality (5):

$$\begin{aligned}
 &\sin \alpha \sin \beta \leq \sin^2 \frac{\alpha + \beta}{2} \\
 \text{and} \\
 &\sin \gamma \sin 60^\circ \leq \sin^2 \frac{\gamma + 60^\circ}{2},
 \end{aligned}$$

and from here after multiplying them:

$$\sin \alpha \sin \beta \sin \gamma \sin 60^\circ \leq \left(\sin \frac{\alpha + \beta}{2} \sin \frac{\gamma + 60^\circ}{2} \right)^2. \quad (6)$$

It follows from the inequality (5), that:

$$\sin \frac{\alpha + \beta}{2} \sin \frac{\gamma + 60^\circ}{2} \leq \sin^2 \frac{\alpha + \beta + \gamma + 60^\circ}{4},$$

and from here:

$$\begin{aligned}
 &\left(\sin \frac{\alpha + \beta}{2} \sin \frac{\gamma + 60^\circ}{2} \right)^2 \leq \sin^4 \frac{\alpha + \beta + \gamma + 60^\circ}{4}, \text{ i.e.} \\
 &\left(\sin \frac{\alpha + \beta}{2} \sin \frac{\gamma + 60^\circ}{2} \right)^2 \leq \sin^4 60^\circ
 \end{aligned}$$

or

$$\left(\sin \frac{\alpha + \beta}{2} \sin \frac{\gamma + 60^\circ}{2} \right)^2 \leq \frac{9}{16}. \quad (7)$$

Inequalities (6) and (7) follow from here:

$$\sin \alpha \sin \beta \sin \gamma \sin 60^\circ \leq \frac{9}{16},$$

and further:

$$\sin \alpha \sin \beta \sin \gamma \leq \frac{\frac{9}{16}}{\frac{\sqrt{3}}{2}}, \text{ i.e.}$$

$$\sin \alpha \sin \beta \sin \gamma \leq \frac{3\sqrt{3}}{8}, \text{ q.e.d.}$$

Note, that equality holds if and only if $\alpha = \beta = \gamma = 60^\circ$ (equilateral triangle).

Now, we will prove the inequality (4).

We deduce from the Heron's formula $F^2 = s(s-a)(s-b)(s-c)$, where $s = \frac{a+b+c}{2}$, a new formula for the area of the $\triangle ABC$:

$$16F^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) - a^4 - b^4 - c^4. \quad (8)$$

After squaring, the inequality (4) turns to be equivalent to the inequality:

$$\begin{aligned} & (a^2 + b^2 + c^2)^2 \geq 3 \cdot 16F^2 \\ & \stackrel{(8)}{\Leftrightarrow} (a^2 + b^2 + c^2)^2 \geq 6(a^2b^2 + b^2c^2 + c^2a^2) - 3(a^4 + b^4 + c^4) \\ & \Leftrightarrow 4(a^4 + b^4 + c^4) - 4(a^2b^2 + b^2c^2 + c^2a^2) \geq 0 : 2 \\ & \Leftrightarrow 2(a^4 + b^4 + c^4) - 2(a^2b^2 + b^2c^2 + c^2a^2) \geq 0 \\ & \Leftrightarrow (a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 \geq 0. \end{aligned} \quad (9)$$

The last is obvious. Thus, the inequality (9) is true and consequently the inequality (4) is also true. The equality in (4) holds if and only if $a = b = c$ (equilateral triangle).

Now, we will give a new proof of the inequality (4).

From the well known inequality:

$$\begin{aligned} & a^2 + b^2 + c^2 \geq ab + bc + ca \\ & \left(\Leftrightarrow \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2] \geq 0 \right), \end{aligned}$$

using the well known formulae of the area of the triangle:

$$F = \frac{ab}{2} \sin \gamma = \frac{bc}{2} \sin \alpha = \frac{ac}{2} \sin \beta,$$

we get:

$$a^2 + b^2 + c^2 \geq 2F \left(\frac{1}{\sin \alpha} + \frac{1}{\sin \beta} + \frac{1}{\sin \gamma} \right). \quad (10)$$

We use the inequality between the arithmetic and the geometric means of three positive numbers:

$$\frac{1}{\sin \alpha} + \frac{1}{\sin \beta} + \frac{1}{\sin \gamma} \geq 3 \sqrt[3]{\frac{1}{\sin \alpha \sin \beta \sin \gamma}},$$

and from here we get the inequality (3):

$$\begin{aligned} \frac{1}{\sin \alpha \sin \beta \sin \gamma} &\geq \frac{8}{3\sqrt{3}} : \\ \frac{1}{\sin \alpha} + \frac{1}{\sin \beta} + \frac{1}{\sin \gamma} &\geq 3 \cdot \frac{2}{\sqrt{3}}, \text{ i.e.} \\ \frac{1}{\sin \alpha} + \frac{1}{\sin \beta} + \frac{1}{\sin \gamma} &\geq 2\sqrt{3}. \end{aligned} \quad (11)$$

Now, the inequality (4) follows from (10) and (11).

Finally, we will prove the inequalities (1) and (2).

Proof (of the inequality (1)):

Since $AH \perp BC$ and $AC \perp BE$ (Fig.1), it follows that $\angle HAE = \angle EBC$ and consequently $\angle AEH = \angle BEC = 90^\circ$. Thus, $\triangle AHE \sim \triangle BEC$. From here we have:

$$\begin{aligned} \frac{AH}{BC} &= \frac{AE}{BE}, \text{ i.e.} \\ \frac{AH}{a} &= \frac{AE}{h_b}. \end{aligned} \quad (12)$$

At the same time the right-angle $\triangle ABE$ gives, that:

$$\begin{aligned} \cos \alpha &= \frac{AE}{AB} = \frac{AE}{c}, \\ \text{and from here by the cosine law:} \\ AE &= c \cdot \cos \alpha = c \cdot \frac{b^2 + c^2 - a^2}{2bc}, \text{ i.e.} \\ AE &= \frac{b^2 + c^2 - a^2}{2b}. \end{aligned} \quad (13)$$

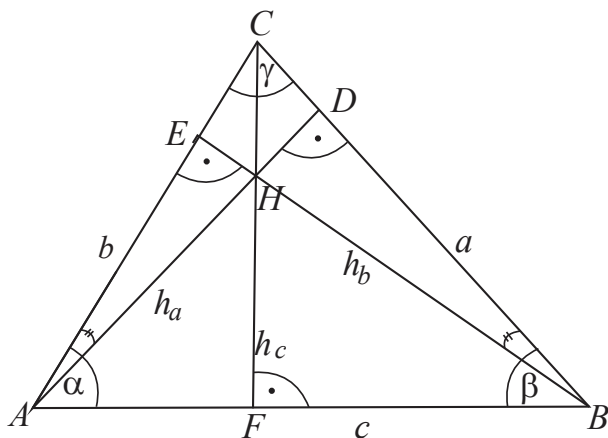


Fig. 1

Because $h_b = \frac{2F}{b}$, we get from (12) and (13), that:

$$\frac{AH}{a} = \frac{b^2 + c^2 - a^2}{4F} \quad (14)$$

Analogously

$$\frac{BH}{b} = \frac{c^2 + a^2 - b^2}{4F} \quad (15)$$

and

$$\frac{CH}{c} = \frac{a^2 + b^2 - c^2}{4F}. \quad (16)$$

Applying (14), (15) and (16), we deduce that:

$$\frac{AH}{a} + \frac{BH}{b} + \frac{CH}{c} = \frac{a^2 + b^2 + c^2}{4F}. \quad (17)$$

Finally, from (17) and (4) we have:

$$\frac{AH}{a} + \frac{BH}{b} + \frac{CH}{c} \geq \sqrt{3}, \text{ q.e.d.}$$

The equality holds if and only if $a=b=c$ (equilateral triangle).

Proof (of the inequality (2)): Feom (12) we have:

$$\frac{AH}{a} = \frac{AE}{h_b} = \text{ctg} \alpha, \text{ i.e.}$$

$$\text{Analogously: } \frac{AH}{a} = \frac{1}{\operatorname{tg} \alpha}. \quad (18)$$

$$\frac{BH}{b} = \frac{1}{\operatorname{tg} \beta} \quad (19)$$

and

$$\frac{CH}{c} = \frac{1}{\operatorname{tg} \gamma}, \quad (20)$$

and from here after multiplying (18), (19) and (20), we obtain:

$$\frac{AH}{a} \cdot \frac{BH}{b} \cdot \frac{CH}{c} = \frac{1}{\operatorname{tg} \alpha \operatorname{tg} \beta \operatorname{tg} \gamma}. \quad (21)$$

Note the well known equality

$$\operatorname{tg} \alpha + \operatorname{tg} \beta + \operatorname{tg} \gamma = \operatorname{tg} \alpha \operatorname{tg} \beta \operatorname{tg} \gamma.$$

It follows by the inequality between the arithmetic and the geometric means for three positive numbers ($\operatorname{tg} \alpha, \operatorname{tg} \beta, \operatorname{tg} \gamma > 0$, because the triangle is acute), that:

$$\begin{aligned} \operatorname{tg} \alpha \operatorname{tg} \beta \operatorname{tg} \gamma &= \operatorname{tg} \alpha + \operatorname{tg} \beta + \operatorname{tg} \gamma \geq 3 \sqrt[3]{\operatorname{tg} \alpha \operatorname{tg} \beta \operatorname{tg} \gamma}^3 \\ &\Leftrightarrow \operatorname{tg}^3 \alpha \operatorname{tg}^3 \beta \operatorname{tg}^3 \gamma \geq 27 \operatorname{tg} \alpha \operatorname{tg} \beta \operatorname{tg} \gamma \\ &\Leftrightarrow (\operatorname{tg} \alpha \operatorname{tg} \beta \operatorname{tg} \gamma)^2 \geq 27 \\ &\Leftrightarrow \operatorname{tg} \alpha \operatorname{tg} \beta \operatorname{tg} \gamma \geq 3\sqrt{3} \\ &\Leftrightarrow \frac{1}{\operatorname{tg} \alpha \operatorname{tg} \beta \operatorname{tg} \gamma} \leq \frac{1}{3\sqrt{3}}. \end{aligned} \quad (22)$$

Finally, from (21) and (22) we obtain the inequality (2), q.e.d. The equality holds if and only if $\alpha = \beta = \gamma = \frac{\pi}{3}$ (equilateral triangle).

REFERENCES

- Arslanagić, Š. (2005). *Matematika za nadarene*. Sarajevo: Bosanska riječ.
 Bottema, O., R. Ž. Đorđević, R. R. Janić, D. S. Mitrinović & P.M. Vasić (1969). *Geometric Inequalities*. Groningen (The Netherlands): Wolters-Noordhoff Publishing.
 Bulajich Manfrino, R., J. A. Gomez Ortega & R. Valdez Delgado (2009). *Inequalities – A Mathematical Olympiad Approach*. Basel•Boston•Berlin: Birkhäuser.

- Grozdev, S. (2007). *For High Achievements in Mathematics. The Bulgarian Experience (Theory and Practice)*. Sofia: ADE.
- Cvetkovski, Z. (2012). *Inequalities – Theorems, Techniques and Selected Problems*. Berlin, Heidelberg: Springer Verlag.
- Sergeeva, T., M. Shabanova & S. Grozdev (2014). *Foundations of Dynamic Geometry*. Moscow: ASOU [Сергеева, Т., М. Шабанова & С. Гроздев (2014). *Основы динамической геометрии*. Москва: АСОУ.]

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