Educational Research
Научно–методически статии

THREE SOLUTIONS OF A PROBLEM WITH FOUR CIRCLES

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Abstract. The paper considers three solutions of an interesting problem with four circles.
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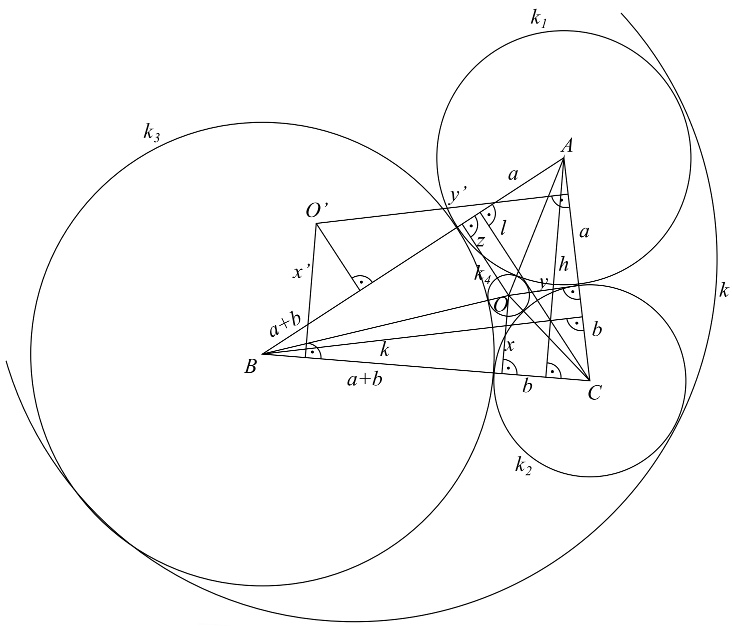
The problem under consideration is the following: *Given are three tangent to each other circles with radii* a, b *and* $a+b$ *respectively, where* $a, b \in \mathbb{R}$ *are positive real numbers. Find the radius of a fourth circle tangent to each of these three circles.*

Solution 1. Let the given circles be denoted as follows: $k_1(A,a)$, $k_2(C,b)$, $k_3(B,a+b)$ and $k_4(O,\rho)$; while $AC = a+b$, $BC = a+2b$, $AB = 2a+b$.



Let h, k, l be the altitudes of ΔABC and let x, y, z be the distances from the point O to the sides BC, AC, AB of ΔABC.

We will use formula (1) from (Adamar, 1957, p. 564):

$$\frac{x}{2\rho}-\frac{h}{2(s-BC)}=1;$$

(where s is the semi-perimeter of ΔABC, i.e. $s=2a+2b$). Analogously,

$$\frac{y}{2\rho}-\frac{k}{2(s-AC)}=1 \text{ and } \frac{z}{2\rho}-\frac{l}{2(s-AB)}=1 \text{, i.e.}$$

$$\frac{x}{2\rho}-\frac{h}{2a}=1;\ \frac{y}{2\rho}-\frac{k}{2(a+b)}=1;\ \frac{z}{2\rho}-\frac{l}{2b}=1,$$

$$\frac{x}{2\rho}-\frac{h}{2a}=1;\ \frac{y}{2\rho}-\frac{k}{2(a+b)}=1;\ \frac{z}{2\rho}-\frac{l}{2b}=1.$$

From here:

$$\frac{x}{h}-\frac{\rho}{a}=\frac{2\rho}{h};\ \frac{y}{k}-\frac{\rho}{a+b}=\frac{2\rho}{k};\ \frac{z}{l}-\frac{\rho}{b}=\frac{2\rho}{l},$$

$$\frac{x}{h}+\frac{y}{k}+\frac{z}{l}=\frac{\frac{x\cdot BC}{2}}{\frac{h\cdot BC}{2}}+\frac{\frac{y\cdot AC}{2}}{\frac{k\cdot AC}{2}}+\frac{\frac{z\cdot AB}{2}}{\frac{l\cdot AB}{2}}=\frac{F_{\Delta OBC}+F_{\Delta OAC}+F_{\Delta OAB}}{F_{\Delta ABC}}=\frac{F_{\Delta ABC}}{F_{\Delta ABC}}=1:$$

(1) $$1=\rho\left(\frac{1}{a}+\frac{1}{a+b}+\frac{1}{b}+\frac{2}{h}+\frac{2}{k}+\frac{2}{l}\right)$$

We have:

$$F\left(=F_{\Delta ABC}\right)=\frac{(a+2b)h}{2}\Rightarrow\frac{2}{h}=\frac{a+2b}{F},$$

$$F\left(=F_{\Delta ABC}\right)=\frac{(a+b)\cdot k}{2}\Rightarrow\frac{2}{k}=\frac{a+b}{F},$$

$$F\left(=F_{\Delta ABC}\right)=\frac{(2a+b)\cdot l}{2}\Rightarrow\frac{2}{l}=\frac{2a+b}{F},$$

where $F=\sqrt{(2a+2b)\cdot a\cdot b\cdot(a+b)}=(a+b)\sqrt{2ab}$.

If follows now from (1), that:

$$\rho=\frac{1}{\frac{1}{a}+\frac{1}{a+b}+\frac{1}{b}+\frac{a+2b}{(a+b)\sqrt{2ab}}+\frac{a+b}{(a+b)\sqrt{2ab}}+\frac{2a+b}{(a+b)\sqrt{2ab}}},$$

$$\rho=\frac{1}{\frac{b(a+b)+ab+a(a+b)}{ab(a+b)}+\frac{a+2b+a+b+2a+b}{\sqrt{2ab}\,(a+b)}},$$

$$\rho=\frac{1}{\frac{a^2+b^2+3ab}{ab(a+b)}+\frac{2\sqrt{2ab}}{ab}}=\frac{1}{\frac{a^2+b^2+3ab+2\sqrt{2ab}\,(a+b)}{ab(a+b)}},$$

$$\rho=\frac{ab(a+b)}{a^2+b^2+3ab+2\sqrt{2ab}\,(a+b)}.$$

Remark 1. For the radius of the circle k' with center O' we get analogously:

$$\rho'=\left|\frac{F}{2s-(r_a+r_b+r_c)}\right|,$$

where $F\left(=F_{\Delta ABC}\right)=(a+b)\sqrt{2ab}$, $2s=4(a+b)$, $r_a=\frac{F}{s-(a+2b)}=\frac{(a+b)\sqrt{2ab}}{a}$,

$r_b=\frac{F}{s-(a+b)}=\frac{(a+b)\sqrt{2ab}}{a+b}$; $r_c=\frac{F}{s-(2a+b)}=\frac{(a+b)\sqrt{2ab}}{b}$.

Solution 2. The semi-perimeter of ΔAOC is $\frac{1}{2}(a+b+a+\rho+b+\rho)=a+b+\rho$ and we have

$$\cos^2\left(\frac{1}{2}\measuredangle AOC\right)=\frac{\rho(a+b+\rho)}{(a+\rho)(b+\rho)};\quad \sin^2\left(\frac{1}{2}\measuredangle AOC\right)=\frac{ab}{(a+\rho)(b+\rho)}.$$

We will use the following well known theorem (Grozdev, 2007):

If $\alpha'+\beta'+\gamma'=180^0$, then

$\sin^2\alpha'-\sin^2\beta'-\sin^2\gamma'+2\sin\beta'\sin\gamma'\cos a'=0$

and from here by $\alpha'=\frac{1}{2}\measuredangle AOC$, $\beta'=\frac{1}{2}\measuredangle BOC$ and $\gamma'=\frac{1}{2}\measuredangle AOB$, we get:

$$\frac{ab}{(a+\rho)(b+\rho)}-\frac{b(a+b)}{(b+\rho)(a+b+\rho)}-\frac{a(a+b)}{(a+\rho)(a+b+\rho)}+2\cdot\frac{(a+b)\sqrt{ab\rho(a+b+\rho)}}{(a+b+\rho)(a+\rho)(b+\rho)}=0\text{, i.e.}$$

$$\frac{a+b+\rho}{a+b}-\frac{a+\rho}{a}-\frac{b+\rho}{b}+2\sqrt{\frac{\rho(a+b+\rho)}{a+b}}=0.$$

Further, we divide by ρ and use the substitutions $\alpha=\frac{1}{a}$, $\beta=\frac{1}{b}$, $\gamma=\frac{1}{a+b}$ and $\delta=\frac{1}{\rho}$:

$$(*)\qquad \alpha-\beta-\gamma-\delta+2\sqrt{\beta\gamma+\gamma\delta+\delta\beta}=0$$

Since

$$(\alpha+\beta+\gamma+\delta)^2=(\alpha-\beta-\gamma-\delta)^2+4(\alpha\beta+\alpha\gamma+\alpha\delta)$$

$$\overset{(*)}{=}4(\beta\gamma+\gamma\delta+\delta\beta)+4(\alpha\beta+\alpha\gamma+\alpha\delta)$$

$$=2(\alpha+\beta+\gamma+\delta)^2-2\left(\alpha^2+\beta^2+\gamma^2+\delta^2\right)$$

$$\Rightarrow 2\left(\alpha^2+\beta^2+\gamma^2+\delta^2\right)=(\alpha+\beta+\gamma+\delta)^2,$$

We obtain:

$$2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{(a+b)^2}+\frac{1}{\rho^2}\right)=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{a+b}+\frac{1}{\rho}\right)^2$$

$$\Leftrightarrow \frac{2}{\rho^2}+2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{(a+b)^2}\right)=\frac{1}{\rho^2}+2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{a+b}\right)\cdot\frac{1}{\rho}+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{a+b}\right)^2$$

$$\Leftrightarrow \frac{1}{\rho^2}-2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{a+b}\right)\cdot\frac{1}{\rho}+2\left(\frac{1}{(a+b)^2}+\frac{1}{a^2}+\frac{1}{b^2}\right)-\left(\frac{1}{a+b}+\frac{1}{a}+\frac{1}{b}\right)^2=0.$$

If $\frac{1}{\rho}=t$, then:

$$\Rightarrow t=\frac{1}{a}+\frac{1}{b}+\frac{1}{a+b}\pm\sqrt{2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{a+b}\right)^2-2\left(\frac{1}{(a+b)^2}+\frac{1}{a^2}+\frac{1}{b^2}\right)}$$

$$\Rightarrow t=\frac{1}{a}+\frac{1}{b}+\frac{1}{a+b}\pm\sqrt{4\left(\frac{1}{ab}+\frac{1}{a(a+b)}+\frac{1}{b(a+b)}\right)}$$

$$\Rightarrow t=\frac{1}{a}+\frac{1}{b}+\frac{1}{a+b}\pm 2\sqrt{\frac{a+b+b+a}{ab(a+b)}}$$

$$\Rightarrow t=\frac{1}{a}+\frac{1}{b}+\frac{1}{a+b}\pm 2\sqrt{\frac{2}{ab}}\ .$$

We have

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{a+b}=\frac{a+b}{ab}+\frac{1}{a+b}\overset{AM-GM}{\geq}2\sqrt{\frac{1}{ab}}\ .$$

Since $2\sqrt{\frac{1}{ab}}<2\sqrt{\frac{2}{ab}}$, then $t=\frac{1}{a}+\frac{1}{b}+\frac{1}{a+b}-2\sqrt{\frac{2}{ab}}$. It follows that there is only one solution:

$$t=\frac{1}{a}+\frac{1}{b}+\frac{1}{a+b}+2\sqrt{\frac{2}{ab}}\text{, i.e.}$$

$$t=\frac{b(a+b)+a(a+b)+ab}{ab(a+b)}+\frac{2\sqrt{2}}{\sqrt{ab}},$$

$$t=\frac{a^2+b^2+3ab}{ab(a+b)}+\frac{2\sqrt{2ab}}{ab},$$

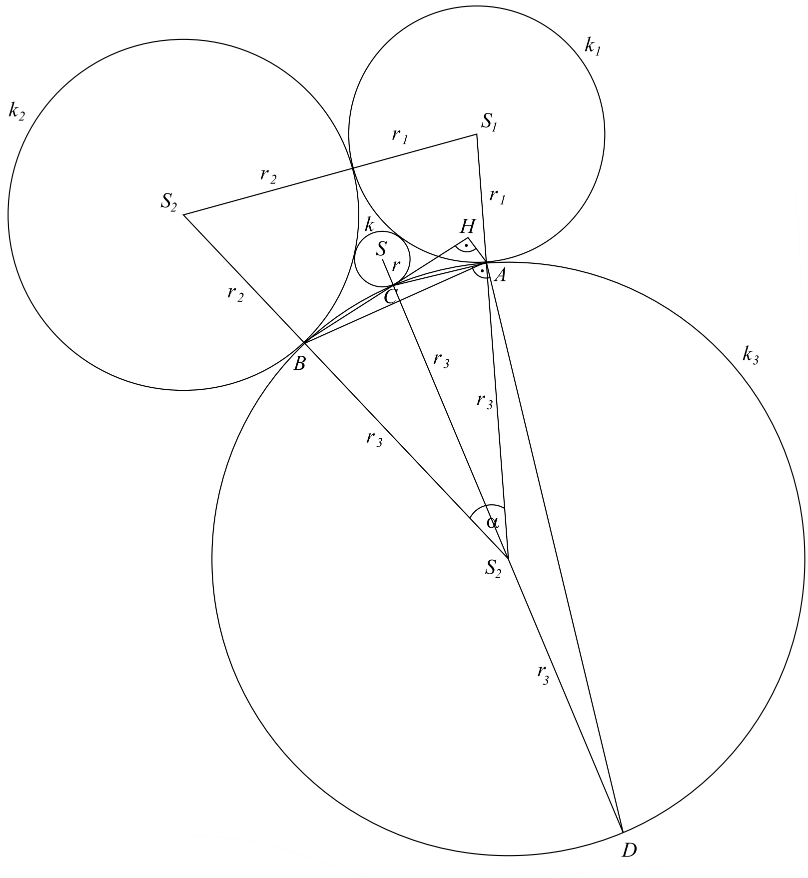
$$t=\frac{a^2+b^2+3ab+2\sqrt{2ab}(a+b)}{ab(a+b)}\text{, i.e.}$$

$$\rho=\frac{ab(a+b)}{a^2+b^2+3ab+2\sqrt{2ab}(a+b)}\ .$$

Solution 3. Let B and A be the tangent points of the circles k_2 and k_3, k_1 and k_3, respectively. By the law of cosines for ΔS_1AB we have:

$$\left(AB\right)^2 = \left(S_3B\right)^2 + \left(S_3A\right)^2 - 2 \cdot S_3B \cdot S_3A\cos\alpha \text{, i.e.}$$

(2) $$\left(AB\right)^2 = 2r_3^2\left(1-\cos\alpha\right).$$



Analogously, by the law of cosines for $\Delta S_1S_2S_3$:

$$\cos\alpha = \frac{\left(S_1S_3\right)^2 + \left(S_2S_3\right)^2 - \left(S_1S_2\right)^2}{2S_1S_3 \cdot S_2S_3} \text{, i.e.}$$

$$\cos\alpha = \frac{\left(r_1+r_3\right)^2 + \left(r_2+r_3\right)^2 - \left(r_1+r_2\right)^2}{2\left(r_1+r_3\right)\left(r_2+r_3\right)}$$

or

$$(3)\qquad \cos\alpha=\frac{r_3^2+r_1r_3+r_2r_3-r_1r_2}{(r_1+r_3)(r_2+r_3)}.$$

Now, it follows from (2) and (3):

$$(AB)^2=2r_3^2\left(1-\frac{r_3^2+r_1r_3+r_2r_3-r_1r_2}{(r_1+r_3)(r_2+r_3)}\right)$$

$$(4)\qquad \Rightarrow (AB)^2=\frac{4r_1r_2r_3^2}{(r_1+r_3)(r_2+r_3)}.$$

Let $k(S,r)$ be the fourth circle tangent to each of the circles k_1, k_2 and k_3. Let A, B and C be the tangent points of circles k_1 and k_3, k_2 and k_3, k_3 and k, respectively. Let the point H be the foot of the normal from the point A to the line BC. Using (4) for the circles k_1,k_2,k_3 firstly, then for the circles k_2,k_3,k and finally for the circles k_1,k_3,k, we get:

$$(AB)^2=\frac{4r_1r_2r_3^2}{(r_1+r_3)(r_2+r_3)},$$

$$(5)\qquad (BC)^2=\frac{4r_2r_3^2r}{(r_2+r_3)(r+r_3)},$$

$$(6)\qquad (AC)^2=\frac{4r_1r_3^2r}{(r_1+r_3)(r+r_3)}.$$

The similarity of the right triangles ΔABH and ΔCDA gives:

$$\frac{AC}{CD}=\frac{AH}{AB},$$

and from here by (4) and (6), wd obtain:

$$(7)\qquad (AH)^2=\frac{(AB)^2\cdot(AC)^2}{4r_3^2}=\frac{4r_1^2r_2r_3^2r}{(r_1+r_3)^2(r_2+r_3)(r+r_3)},$$

$$(8)\qquad (CH)^2=(AC)^2-(AH)^2=\frac{4r_1r_3^2r}{(r_1+r_3)(r+r_3)}\left(1-\frac{r_1r_2}{(r_1+r_3)(r_2+r_3)}\right).$$

By (4), (5), (6), (7) and (8):

$$(AB)^2 = (BH)^2 + (AH)^2 = (BC + CH)^2 + (AH)^2 = (BC)^2 + 2BC \cdot CH + (CH)^2 + (AH)^2 =$$

$= (AC)^2 + (BC)^2 + 2BC \cdot CH$, i.e.

$$\frac{4r_1r_2r_3^2}{(r_1+r_3)(r_2+r_3)} = \frac{4r_1r_3^2 r}{(r_1+r_3)(r+r_3)} + \frac{4r_2r_3^2 r}{(r_2+r_3)(r+r_3)} + 2 \cdot \frac{2r_3\sqrt{r_2 r}}{\sqrt{(r_2+r_3)(r+r_3)}} \cdot$$

$$\cdot \frac{2r_3\sqrt{r_1 r}}{\sqrt{(r_1+r_3)(r+r_3)}} \sqrt{1 - \frac{r_1r_2}{(r_1+r_3)(r_2+r_3)}} \, .$$

We get: $r_1r_2(r+r_3) = r_1r(r_2+r_3) + r_2r(r_1+r_3) + 2r\sqrt{r_1r_2r_3(r_1+r_2+r_3)}$

and finally $r = \dfrac{r_1r_2r_3}{r_1r_2 + r_2r_3 + r_1r_3 + 2\sqrt{r_1r_2r_3(r_1+r_2+r_3)}}$.

If $r_1 = a,\ r_2 = b,\ r_3 = a+b$, then:

$$r = \frac{ab(a+b)}{ab + b(a+b) + a(a+b) + 2\sqrt{ab(a+b)(2a+2b)}} \text{, i.e.}$$

$$r = \frac{ab(a+b)}{a^2 + b^2 + 3ab + 2(a+b)\sqrt{2ab}} \, .$$

REFERENCES

1. Adamar, Ž. (1957). *Elementarnaja geometrija, Knjiga 1., Planimetrija*. Moskva.
2. Grozdev, S. (2007). *For High Achievements in Mathematics. The Bulgarian*
3. *Experience (Theory and Practice)*. Sofia: ADE. (ISBN 978-954-92139-1-1), 295 pages.

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