

THREE INEQUALITIES FOR THE ALTITUDES OF A TRIANGLE

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Abstract. The following three inequalities are considered in the paper

$$h_a + h_b + h_c \leq 4R + r, \quad h_a + h_b + h_c \leq \frac{a^2 + b^2 + c^2}{2R}, \quad h_a + h_b + h_c \leq \frac{2s^2}{3R}.$$

Arrangements of inequalities are considered too.

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We will prove, that:

$$h_a + h_b + h_c \leq 4R + r, \quad (1)$$

$$h_a + h_b + h_c \leq \frac{a^2 + b^2 + c^2}{2R}, \quad (2)$$

$$h_a + h_b + h_c \leq \frac{2s^2}{3R}, \quad (3)$$

where a, b, c are the sides of a triangle $\triangle ABC$; h_a, h_b, h_c are the corresponding altitudes; s is the semiperimeter and R and r are the circumradius and the inradius of the triangle. For the proof of (1) we will prove the inequality:

$$4R + r \geq s\sqrt{3}. \quad (4)$$

What follows is the proof of (4).

Proof: We will use the well-known formulae (see (Grozdev, 2007)):

$$r_a = \frac{F}{s-a}, \quad r_b = \frac{F}{s-b}, \quad r_c = \frac{F}{s-c},$$

where r_a, r_b, r_c are the exradii of $\triangle ABC$ and F is its area.

Using that $F = rs$ and applying the Heron's formula $F = \sqrt{s(s-a)(s-b)(s-c)}$ we get:

$$r_a r_b = \frac{F^2}{(s-a)(s-b)} = \frac{s(s-a)(s-b)(s-c)}{(s-a)(s-b)} = s(s-c),$$

Analogously, $r_b r_c = s(s-a)$ and $r_a r_c = s(s-b)$. Consequently:

$$r_a r_b + r_b r_c + r_a r_c = s(s-a) + s(s-b) + s(s-c) = s(s-a+s-b+s-c) = s(3s-2s) = s^2, \text{ i.e.}$$

$$r_a r_b + r_b r_c + r_a r_c = s^2. \quad (5)$$

Further, apply the well-known formula $abc = 4RF$:

$$\begin{aligned} r_a + r_b + r_c - r &= \frac{F}{s-a} + \frac{F}{s-b} + \frac{F}{s-c} - \frac{F}{s} = \\ &= F \left[\frac{s-b+s-a}{(s-a)(s-b)} + \frac{s-s+c}{s(s-c)} \right] = F \left[\frac{2s-(a+b)}{(s-a)(s-b)} + \frac{c}{s(s-c)} \right] = \\ &= F \left[\frac{c}{(s-a)(s-b)} + \frac{c}{s(s-c)} \right] = cF \cdot \frac{s(s-c) + (s-a)(s-b)}{s(s-a)(s-b)(s-c)} = \\ &= cF \cdot \frac{2s^2 - (a+b+c)s + ab}{F^2} = c \cdot \frac{ab}{F} = \frac{abc}{F} = \frac{4RF}{F} = 4R, \text{ i.e.} \end{aligned}$$

$$r_a + r_b + r_c = 4R + r. \quad (6)$$

Now, we go back to the inequality (4). Apply the following evident inequality:

$$(r_a + r_b + r_c)^2 \geq 3(r_a r_b + r_b r_c + r_a r_c)$$

$$\left(\Leftrightarrow \frac{1}{2} \left[(r_a - r_b)^2 + (r_b - r_c)^2 + (r_c - r_a)^2 \right] \geq 0 \right).$$

It follows from the equalities (5) and (6) that:

$$(4R + r)^2 \geq 3 \cdot s^2,$$

and from here:

$$4R + r \geq s\sqrt{3}, \text{ q.e.d.}$$

An equality holds in (4) iff $\triangle ABC$ is equilateral.

For the proof of the inequality (1) we will use also the well-known Euler's inequality:

$$R \geq 2r . \quad (7)$$

(Different proofs of this inequality could be found in (Arslanagić, 2005), (Arslanagić, 2008) and (Arslanagić, 2009).)

Also, we will use the following well-known equality:

$$ab + bc + ca = s^2 + r^2 + 4Rr , \quad (8)$$

the proof of which could be found in (Arslanagić, 2005) too.

We have from (8) that:

$$\begin{aligned} h_a + h_b + h_c &= \frac{2F}{a} + \frac{2F}{b} + \frac{2F}{c} = 2F \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = \\ &= 2 \cdot \frac{abc}{4R} \cdot \frac{ab + bc + ca}{abc} = \frac{s^2 + r^2 + 4Rr}{2R} , \text{ i.e.} \\ h_a + h_b + h_c &= \frac{s^2 + r^2 + 4Rr}{2R} . \end{aligned} \quad (9)$$

Now, using the inequality (4) we get:

$$\begin{aligned} s\sqrt{3} &\leq 4R + r \\ \Leftrightarrow 3s^2 &\leq 16R^2 + 8Rr + r^2 , \end{aligned} \quad (10)$$

and applying the inequality (7), we have:

$$\begin{aligned} 24R^2 - 6Rr - 3r^2 - (16R^2 + 8Rr + r^2) &= \\ = 8R^2 - 14Rr - 4r^2 &= 2(4R^2 - 7Rr - 2r^2) = \\ = 2(4R + r)(R - 2r) &\geq 0 , \text{ i.e.} \\ 16R^2 + 8Rr + r^2 &\leq 24R^2 - 6Rr - 3r^2 . \end{aligned} \quad (11)$$

It follows from (10) and (11) that:

$$\begin{aligned} 3s^2 &\leq 24R^2 - 6Rr - 3r^2 \\ \Leftrightarrow s^2 &\leq 8R^2 - 2Rr - r^2 . \end{aligned} \quad (12)$$

Finally, from (9) and (12) we get:

$$h_a + h_b + h_c = \frac{s^2 + r^2 + 4Rr}{2R} \leq \frac{8R^2 - 2Rr - r^2 + r^2 + 4Rr}{2R} , \text{ i.e.}$$

$$h_a + h_b + h_c \leq 4R + r,$$

and this is the inequality (1), q.e.d.

An equality holds in (4) iff $h_a = h_b = h_c \Rightarrow a = b = c$, i.e. when $\triangle ABC$ is equilateral.

Remark 1: We will prove the following generalization of the inequality (1):

$$h_a^\alpha + h_b^\alpha + h_c^\alpha \leq 3^{1-\alpha} (4R + r)^\alpha, \quad (13)$$

where $\alpha \in [0, 1]$ is real number.

For the purpose we consider the function $f(x) = x^\alpha$, where $x > 0$ and $\alpha \in [0, 1]$. Since $f''(x) = \alpha(\alpha - 1)x^{\alpha-2} \leq 0$, this function is concave. Applying Jensen's inequality and the inequality (1) we get:

$$h_a^\alpha + h_b^\alpha + h_c^\alpha \leq 3 \left[\frac{1}{3} (h_a + h_b + h_c) \right]^\alpha \leq 3 \left[\frac{1}{3} (4R + r) \right]^\alpha = 3^{1-\alpha} (4R + r)^\alpha, \text{ q.e.d.}$$

For $\alpha = 1$ we deduce the inequality (1) as a consequence of inequality (13).

In the sequel we deal with the proof of the inequality (2). We have

$$\begin{aligned} h_a + h_b + h_c &= \frac{2F}{a} + \frac{2F}{b} + \frac{2F}{c} = 2F \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = \\ &= 2F \cdot \frac{ab + bc + ca}{abc} = \frac{2abc}{4R} \cdot \frac{ab + bc + ca}{abc} = \frac{ab + bc + ca}{2R}, \end{aligned}$$

and from the well-known inequality

$$ab + bc + ca \leq a^2 + b^2 + c^2 \left(\Leftrightarrow \frac{1}{2} [(a-b)^2 + (b-c)^2 + (c-a)^2] \geq 0 \right) \text{ we obtain:}$$

$$h_a + h_b + h_c \leq \frac{a^2 + b^2 + c^2}{2R}, \text{ q.e.d.}$$

An equality holds in (2) iff $a = b = c$, i.e. when $\triangle ABC$ is equilateral.

It remains to prove the inequality (3). We have:

$$h_a + h_b + h_c = \frac{2F}{a} + \frac{2F}{b} + \frac{2F}{c} = 2F \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = 2F \cdot \frac{ab + bc + ca}{abc}.$$

Thus, it is enough to prove that:

$$2F \cdot \frac{ab + bc + ca}{abc} \leq \frac{2s^2}{3R}$$

$$\Leftrightarrow 3FR \cdot \frac{ab+bc+ca}{4FR} \leq s^2$$

$$\Leftrightarrow 3(ab+bc+ca) \leq 4s^2$$

$$\Leftrightarrow 3(ab+bc+ca) \leq 4 \cdot \frac{(a+b+c)^2}{4}$$

$$\Leftrightarrow 3(ab+bc+ca) \leq (a+b+c)^2$$

$$\Leftrightarrow a^2 + b^2 + c^2 \geq ab + bc + ca$$

$$\Leftrightarrow \frac{1}{2} \left[(a-b)^2 + (b-c)^2 + (c-a)^2 \right] \geq 0.$$

Since the last inequality is obvious, we conclude that the inequality (3) is proved. An equality holds in (3) iff $a = b = c$, i.e. when $\triangle ABC$ is equilateral.

Now we will prove that the inequality (3) is stronger than the inequality (1). It is enough for the purpose to prove the inequality:

$$s^2 \leq 4R^2 + 4Rr + 3r^2, \quad (14)$$

which is known in the mathematical literature as Gerretsen's inequality (see the paper of the present author in the same issue).

We will use the formula for the distance between the incentre and the orthocentre of $\triangle ABC$:

$$|IH|^2 = 2r^2 - 4R^2 \cos \alpha \cos \beta \cos \gamma, \quad (15)$$

where α, β, γ are the angles of the triangle. (Two proofs of this formula could be found in (Arslanagić, 2005).) From the well-known identity for the angles α, β, γ of $\triangle ABC$

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2(1 + \cos \alpha \cos \beta \cos \gamma),$$

we obtain:

$$\cos \alpha \cos \beta \cos \gamma = \frac{1}{2} (\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma) - 1.$$

Using the equality $a^2 + b^2 + c^2 = 2(s^2 + r^2 - 4Rr)$ it follows by the sine law that:

$$\cos \alpha \cos \beta \cos \gamma = \frac{I}{8R^2}(a^2 + b^2 + c^2) - I, \text{ i.e.}$$

$$\cos \alpha \cos \beta \cos \gamma = \frac{I}{4R^2}(s^2 - r^2 - 4Rr) - I, \text{ i.e.}$$

$$4R^2 \cos \alpha \cos \beta \cos \gamma = s^2 - r^2 - 4Rr - 4R^2. \quad (16)$$

From (15) and (16) we obtain:

$$|IH|^2 = 2r^2 - (s^2 - r^2 - 4Rr - 4R^2)$$

and from here (because $|IH|^2 \geq 0$) it follows that

$$2r^2 - (s^2 - r^2 - 4Rr - 4R^2) \geq 0, \text{ i.e.}$$

$$s^2 \leq 4R^2 + 4Rr + 3r^2,$$

which is the inequality (14), q.e.d.

An equality holds in (14) iff $\alpha = \beta = \gamma = \frac{\pi}{3} \Rightarrow a = b = c$, i.e. when $\triangle ABC$ is equilateral.

Further we will prove the inequality:

$$\frac{a^2 + b^2 + c^2}{2R} \leq 4R + r \quad (17)$$

$$\Leftrightarrow a^2 + b^2 + c^2 \leq 2R(4R + r)$$

and by $a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$ we have:

$$2(s^2 - r^2 - 4Rr) \leq 2R(4R + r)$$

$$\Leftrightarrow s^2 \leq 4R^2 + 5Rr + r^2. \quad (18)$$

Since

$$4R^2 + 4Rr + 3r^2 \leq 4R^2 + 5Rr + r^2, \quad (19)$$

$$\Leftrightarrow Rr \geq 2r^2$$

$$\Leftrightarrow R \geq r \text{ (Euler's inequality),}$$

We conclude easily that the inequality (2) is stronger than the inequality (1), q.e.d.

Finally we will prove that the inequality (3) is stronger than the inequality (2), i.e. that it is valid the following inequality:

$$\frac{2s^2}{3R} \leq \frac{a^2 + b^2 + c^2}{2R} \quad (20)$$

$$\Leftrightarrow 4s^2 \leq 3(a^2 + b^2 + c^2)$$

$$\Leftrightarrow 4 \cdot \frac{(a+b+c)^2}{4} \leq 3(a^2 + b^2 + c^2)$$

$$\Leftrightarrow (a+b+c)^2 \leq 3(a^2 + b^2 + c^2)$$

$$\Leftrightarrow a^2 + b^2 + c^2 \geq ab + bc + ca$$

$$\Leftrightarrow \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2] \geq 0,$$

The last inequality is true and this ends the proof of the assertion that the inequality (20) is true, i.e. that the inequality (3) is stronger than inequality (2). Consequently we have:

$$h_a + h_b + h_c \leq \frac{2s^2}{3R} \leq \frac{a^2 + b^2 + c^2}{2R} \leq 4R + r.$$

More inequalities concerning the altitudes of $\triangle ABC$ could be found in (Bottema et al., 1969), namely:

$$h_a + h_b + h_c \leq s\sqrt{3}, \quad (6.1, \text{ p. 60})$$

$$h_a + h_b + h_c \leq \frac{9}{2}R, \quad (6.10, \text{ p. 62})$$

$$h_a + h_b + h_c \leq 3(R+r), \quad (6.11, \text{ p. 62})$$

$$h_a + h_b + h_c \leq 2R + 5r, \quad (6.12, \text{ p. 62})$$

$$h_a + h_b + h_c \leq \frac{2(R+r)^2}{R}. \quad (6.13, \text{ p. 63}).$$

Including the already proved inequalities (1), (2) and (3) we obtain the following series of inequalities:

$$h_a + h_b + h_c \leq \frac{2s^2}{3R} \leq \frac{2(R+r)^2}{R} \leq 2R + 5r \leq \frac{a^2 + b^2 + c^2}{2R} \leq s\sqrt{3} \leq 3(R+r) \leq 4R + r \leq \frac{9}{2}R.$$

In the series equalities hold iff and only if $\triangle ABC$ is equilateral.

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