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THE PYTHAGOREAN TREE WITH GEOGEBRA AS A MOTIVATOR FOR THE IMPLEMENTATION OF AUGMENTED REALITY IN SCHOOL

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Abstract. A variant for interactive presentation of content with the inclusion of augmented reality using dynamic software GeoGebra is described. A concretization is made with the Pythagorean Tree fractal. The goal is to support the understanding and memorization of the relevant content by providing conditions for research and experience, as well as for gaining experience in using augmented reality as a technology and motivating teachers to implement it in school. Models are described with which the growth of the fractal is observed through animation, as well as changes depending on several parameters. Basic ideas for creating the relevant compositions with dynamic software GeoGebra are described. The goal is to facilitate the use of this resource when working in STEAM centers. An expert assessment of the proposed resources and methodology is presented, as well as the results of a pilot study. Possibilities for continuing research on the topic are discussed both in terms of interpretations related to the fractal (for example, using different angles in the steps, different coloring options, using figures other than regular polygons, etc.), and in terms of software products for creation and technological tools for impact (for example, virtual reality, mapping, etc.). The possibility of using such resources in science museums is also emphasized, which will improve the implementation of their educational function.

Keywords: interactivity; augmented reality; fractal; Pythagorean Tree; STEAM; GeoGebra

1. Introduction

The Pythagorean Theorem is part of the school mathematics content that leaves a lasting mark. This is due to both its numerous applications and the many visual proofs that develop the imagination of students. When using specialized mathematical software in recent years, some of the famous visual proofs of the Pythagorean Theorem have been animated and enriched with numerical information, which continues to increase interest in it. There are such examples in the Virtual Mathematics Laboratory, developed at the Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences, which are used to formulate hypotheses or to discover the idea of proof. Mechanical models are also available, especially in science centers and museums for students, which can be supported with digital models.

A suitable topic for study in STEAM centers or in informal education is "The Pythagorean Tree Fractal". Research on the Pythagorean Tree fractal continues, including algorithms and programming environments for its creation, tools for finding its dimensions, and its relationship with other fractals (Beck et al. 2014), (Browne 2007), (Burch et al. 2022). Educational resources related to the Pythagorean Tree fractal have been developed for different age groups. Various tools have been used to construct the fractal, such as hand-sketching, origami, and computer environments (Spreafico & Tramuns 2019). Commonly, in the developed resources on the subject, it is used or starts with a right-angled isosceles triangle, on the outside of which squares are built (Ghosh 2019). There are studies in which the angles of the initial right-angled triangle are changed, a specific right-angled triangle is considered (Teia 2016), or the squares are replaced with other figures, for example, "moons" (Nordheimer 2010). Emphasis is placed on the areas and perimeters of the respective iterations, as well as on the Pythagorean theorem, angles, congruence of triangles, and geometric sequences (Rezende et al. 2018).

2. Context

Augmented Reality is a technology that improves and facilitates work in various professional and scientific fields. Key features of augmented reality are the combination of real and virtual elements, real-time interaction, and accurate registration of 3D virtual and real objects (Lebamovski & Nikolova 2024), (Azuma 1997), including through the use of digital devices that register pre-set markers (Chotrov et al. 2011), (Maleshkov et al. 2011).

GeoGebra's Augmented Reality mobile app for iPhone and iPad allows students to observe and explore virtual objects in their real-world environment.

The widespread use of GeoGebra in mathematics education has led many teachers and researchers to conduct studies with the app. The results of these studies show that the use of augmented reality has a positive effect on mathematics and science education (Arymbekov et al. 2024), (Bedewy et al. 2021), (del Cerro Velázquez & Morales Méndez 2021), (Hohenwarter et al. 2009), (Iparraguirre-Villanueva et al. 2024), (Lainufar et al. 2021), (Teichrew & Erb 2020), (Tomaschko & Hohenwarter 2020), (Widada et al. 2021). The research is most often related to the teaching of stereometry, to the study of objects from the environment through their modeling, to the development of spatial intelligence, and to the preparation of teachers for the application of technology.

Here we will describe an opportunity for interactive presentation of the Pythagorean Tree fractal, using the possibilities for applying augmented reality through the GeoGebra mobile application1 for Augmented Reality.

3. Scientific apparatus

The classic visual proof of the Pythagorean Theorem is associated with the so-called "Pythagorean underpants" – an image of a right-angled triangle, on the outside of which squares are built. The sum of the areas of the squares on the legs is equal to the area of the square built on the hypotenuse. In the first model of the Pythagorean Tree, squares are used. The growth of the tree can be observed, i.e., the steps when obtaining several consecutive iterations of the fractal (fig. 1). For this purpose, a slider-parameter n is used, with which the individual iterations are shown or hidden.

Initially, the right triangle is isosceles and a symmetrical fractal is constructed accordingly, but the angle of the right triangle can be changed, and animation can be organized. For this purpose, when constructing the fractal, a slider-parameter for the angle is used, defining the specific right triangle (fig. 2).

The size of the figure can also be easily changed, for which there is numerical information.

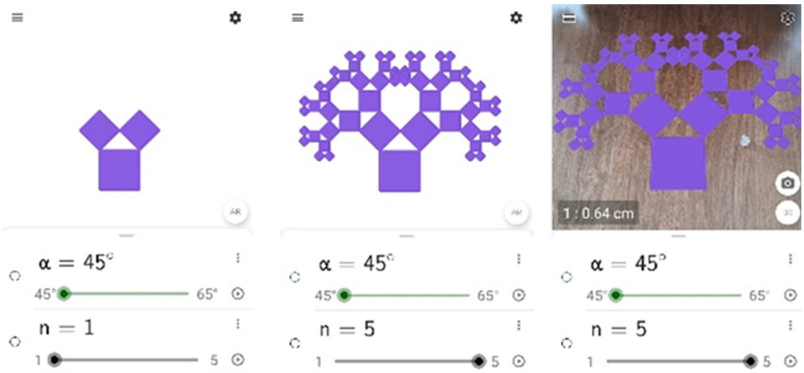

Figure 1. Fractal "Pythagorean Tree" (square). Koya Chehlarova[2], PTAR-4a[3]

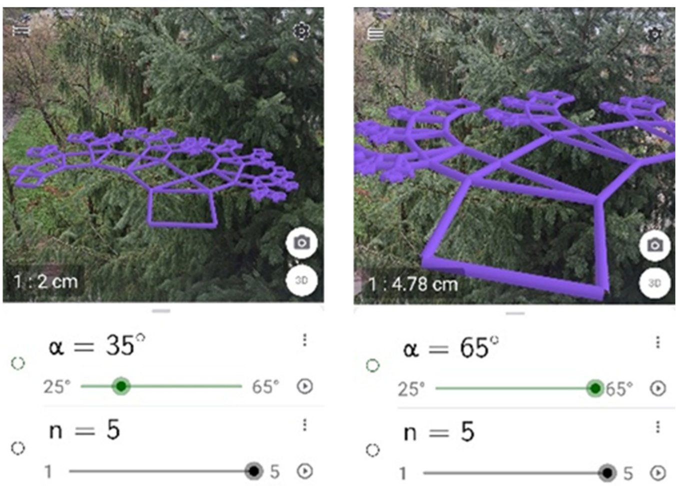

Figure 2. Changing the angle of the "Pythagorean Tree". PTAR 4b[4]

For the construction of the two models presented, which differ in the degree of transparency of the squares, a tool has been prepared in advance, in which after specifying two points and an angle, two squares are constructed, corresponding to the hypotenuses of a right triangle defined by these three elements.

Fig. 3 shows two results: at points A and B with an angle of 45 degrees, and at points C and D with an angle of 65 degrees.

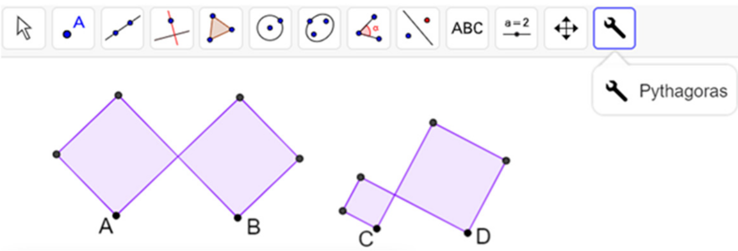

Figure 3. Tool for creating the "Pythagorean Tree" Fractal (square)

The reason for creating this tool is that a characteristic of the fractal is self-similarity, and it is necessary to perform the same construction repeatedly to obtain the corresponding iterations.

In the following models, other regular polygons are used instead of a square. They also achieve a visual proof of the Pythagorean Theorem; respectively, we consider them as Pythagorean Trees. Fig. 4 shows the steps from selecting the Pythagorean tree file with regular pentagons to using augmented reality to find an interior solution.

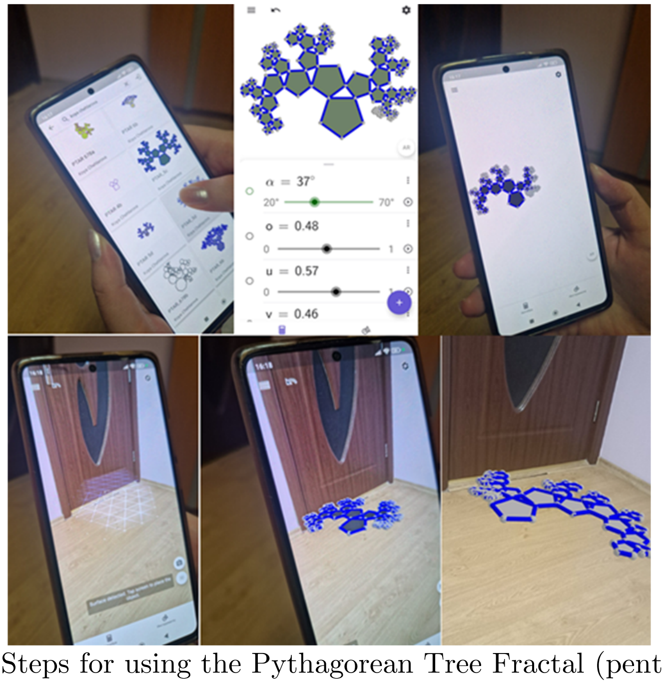

Figure 4. Steps for using the Pythagorean Tree Fractal (pentagon) with augmented reality. PTAR 5c[5]

To achieve the animation in relation to the angle, here again a parameter is used, which changes at a suitable interval according to the particular case. To build these fractals, we have created a tool where, in addition to the two points and the angle, the number of sides of a regular polygon must be specified. To obtain the image in fig. 5 with this tool, the number of sides is 5, and the angle is 45 degrees.

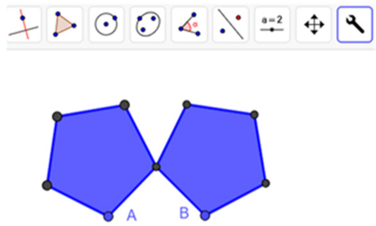

Figure 5. "Pythagorean Tree" Fractal Creation Tool (Regular Polygon)

The images in fig. 6 are a screenshot in 3D mode, a screenshot in AR mode, and a photo with another device in place in the environment where the observation is being performed.

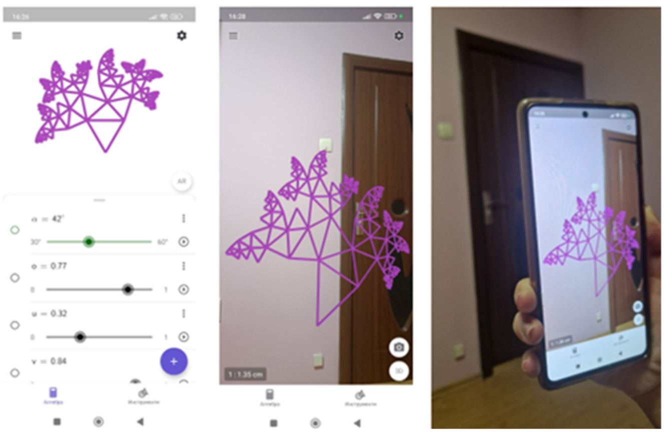

Figure 6. Fractal "Pythagorean Tree" (equilateral triangle). PTAR 3b[6]

Three of the sliders are for providing color changes, which can also be organized dynamically, i.e., animation can be provided in terms of both shape and color.

4. Results

An anonymous survey was conducted with 5 experts. Google Form was used to conduct it. Respondents were informed that completing the form was anonymous and constituted consent to the use of the data for scientific purposes. The age profile of the surveyed experts is shown in fig. 7. The surveyed experts are from the fields of mathematics, information technology, administration, and management and arts. They have different experience in using augmented reality, with options from 1 (low) to 5 (high).

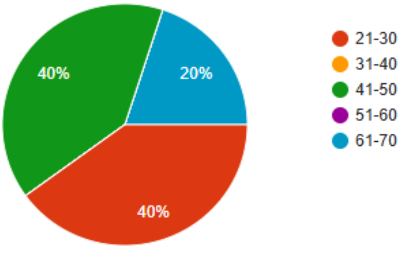

Figure 7. Age characteristics of the surveyed experts

The self-assessments are 1 respondent with a self-assessment of 1, two with a self-assessment of 4, two with a self-assessment of 5. The high self-assessment is for respondents up to 30 years of age.

The results of the evaluations of the presented materials for installing the relevant GeoGebra application for using augmented reality and working with the considered Pythagorean Tree files according to the criteria: easy orientation for installing the application, easy orientation for opening a file, design, motivation for understanding and action, entertainment, usefulness, potential for project work, and overall satisfaction with the work are presented in fig. 8. Expert assessment of the reviewed educational resources is high. All experts gave the highest rating for motivation for understanding and action, entertainment and usefulness. The lowest rating was for ease of orientation for installing the application.

The high scores are also confirmed by the free answers. To the question “Would you share such tasks with others?”, all surveyed experts answered “Yes”. 100% also answered “Yes” to the question “Would you compose such tasks? Why?”, and there are also comments. A participant who has no experience with augmented reality shared “Yes, I would compose such content because using augmented reality in GeoGebra makes learning easier to perceive, visual, interactive and engaging. Through augmented reality, students can see various geometric constructions, figures, graphs and other mathematical objects as part of the real world around them, which facilitates the understanding of abstract concepts and stimulates interest in deeper research and learning.” The arts expert's comment is also aimed at improving the experience of students and users in general: “In the arts, there is a need to implement technology and present cultural content in an interactive and interesting way. This application would be suitable for implementing projects related to the dissemination of art and culture.”

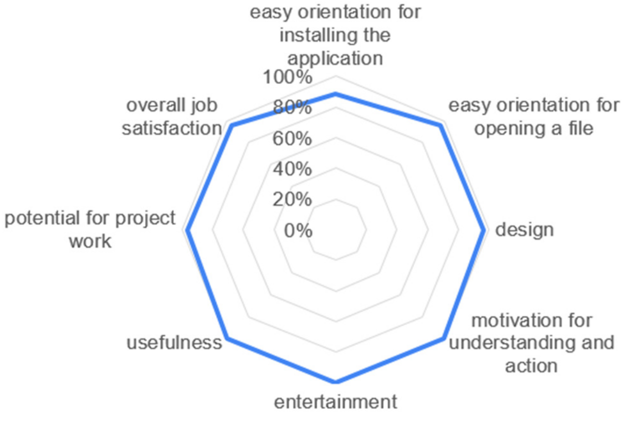

Figure 8. Expert assessment

In opinions, notes and recommendations, respondents described both the specifics of installing and working with the resources, as well as general assessments and expectations for the effect of increasing engagement, developing students' thinking, research skills and creativity: “I believe that using GeoGebra and the possibility of integrating augmented reality in teaching on the topic of Pythagoras will be extremely interesting and useful for students. This technology provides an opportunity for three-dimensional study of mathematical objects from all sides, which significantly facilitates understanding and creating a visual representation of complex concepts and the relationships between them. Thanks to the high level of visualization of the educational content through augmented reality, students will be able not only to see the objects in the real world around them, but also to actively interact with them, which will lead to a deeper understanding and perception of the material. This technology will further motivate students, increase their engagement and interest in mathematics and science in general. The inclusion of augmented reality in the learning units will create conditions for active learning through discovery and exploration, while contributing to the development of critical thinking and creativity. I believe that this technology can be systematically integrated into the learning process by selecting appropriate learning content and creating special tasks that encourage independent research and application of knowledge in different situations."; "The content loads quickly and is easily accessible. There is an option to change the graphics and objects. The files also serve as an introduction to creating graphics and images ourselves"; "Needs more guidance"; "Within 3 minutes: I installed the application, opened 1 file when searching by name with augmented reality; I entered the selected file in the author's GeoGebra profile via the QR code – there was a button below to open via the application and after pressing it directly opened it again via the 3D Calculator. I opened it again in augmented reality mode and took pictures. It took me at least 15 seconds to catch a grid in 3D mode, but this depends on the light and objects in the surrounding space. Dynamic files are very effective - they can be adjusted using the sliders in the application before activating the augmented reality mode."

The emphasis in the free answers of the surveyed experts is both on the motivational and developmental possibilities of the evaluated content, as well as on some technical features. It is not surprising that some of the comments evaluated the time to install and use, given that time is an important indicator when evaluating the effectiveness of implementing new content, methodology, technology in education.

Within the framework of a qualification course with 40 teachers from a secondary school in Bulgaria on the topic of "Technologies for STEAM Education", the considered content was included in order to motivate them to implement augmented reality in school. None of the participants in the qualification course had used augmented reality before. Of these, 37% successfully completed the installation and work with the proposed resources; the rest only observed and commented. 30% managed to install and view the resources independently, while the other 70% required mutual assistance. The implementation took place in 4 to 8 minutes. This was followed by discussions about the independent creation of such resources and the opportunities for creative activity for both teachers and students when using this technology and all participants in the training actively participated in these discussions. Considering the three established stages in the formation of competence for using dynamic software in mathematics education (working with ready-made materials, forming competence for modifying ready-made files and forming competence for independently creating files using the relevant software); these three stages were also outlined for augmented reality.

5. Conclusions

The inclusion of new technologies, like virtual reality, augmented reality, 3D printing, robotic systems, dynamic software, holograms, mapping etc., in the learning process creates conditions for both motivating students and learning through action, as well as for forming competence in using the relevant technologies (Kovatcheva & Koleva 2021), (Chehlarova 2024), (Dimitrova et al. 2025). The main advantage of implementing the described didactic model for providing interactive materials for observation and research, for formulating hypotheses and creating works of art, is the ability to manage the object through the relevant parameters and tools when using a mobile device, in a real environment. Another advantage is the ability to use the widely distributed worldwide product for mathematical education: GeoGebra. A disadvantage of the application of the described didactic model is the need for the mobile application to be installed on the respective mobile device. Given the rapid development of technologies and the frequent updating of mobile applications, the need to edit the relevant educational materials, especially those related to the initial orientation for their use, is also increasing. Considering the strengthening of the educational effect in providing an experience, we believe that in museums it is appropriate to use augmented reality of the type under consideration through paintings, from which a prepared environment for observation and research is obtained.

The acquisition of a specific technology by teachers can be achieved through independent training or within the framework of training related to another specific topic that involves the application of the relevant technology. In the pilot study, teachers were provided with both specific content and the specific technology within an appropriate qualification topic.

The experts surveyed gave the highest marks to motivation for understanding and action, entertainment and usefulness. This result is also confirmed by their free answers related to the survey. Both the expert assessments and the pilot study confirmed the motivating importance of the proposed content for the implementation of augmented reality in school.

6. Future work

The topic is suitable for consideration in the context of STEAM education, as its role in mathematics education has been established in motivating students to perform mathematical activities and guiding them towards choosing majors in the field of mathematics (Karashtranova et al. 2024). The experience can be realized in STEAM centers or in science museums. Considering the understanding of STEAM in (Chehlarova 2024), when studying the fractal "Pythagorean Tree" using information technologies, including in the context of art, three areas are covered – mathematics, technology and art, i.e. STEAM[3] is realized.

It is appropriate to continue the research with an interactive presentation of models of interpretations of this fractal – for example, using different angles at the iterations, different coloring options, different figures at each of the iterations, models that approximate reality etc.

Acknowledgements

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NOTES

1. GeoGebra 3D Calculator https://www.geogebra.org/3d
2. https://www.geogebra.org/u/koy4e6
3. https://www.geogebra.org/m/am5nwsnx
4. https://www.geogebra.org/m/dvtteuxv
5. https://www.geogebra.org/m/r7k2kksb
6. https://www.geogebra.org/m/wczd5emc

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