SEVERAL PROOFS OF AN ALGEBRAIC INEQUALITY

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Abstract. A recently published algebraic inequality is discussed and three new proofs of it are proposed. The paper is of methodological character.

Keywords: inequality, problem solving, proof.

In the Bulgarian journal „Matematika plus“, vol. 20 (80), Nr. 4, 2012 the next problem M+463 (by Jonuc Nedelku and Konstantin Šeu from Ploešti, Romania) is publshed:

Problem. If $x>0,\ y>0,\ z>0$ and $xyz=1$, prove that:

$$\frac{x+y}{1+xy}+\frac{y+z}{1+yz}+\frac{z+x}{1+zx}\geq 3. \qquad (1)$$

In the sequel three proofs of this inequality are proposed.

Proof 1: After transformation (multiplying both sides by $(1+xy)(1+yz)(1+zx)>0$), the given inequality takes the form:

$$x^2z+yz^2+xy^2+x^2y+y^2z+xz^2\geq x+y+z+xy+yz+zx. \qquad (2)$$

The following inequalities are valid:

$$x^2z+yz^2+xy^2\geq x+y+z \qquad (3)$$

and

$$x^2y+y^2z+xz^2\geq xy+yz+zx. \qquad (4)$$

To prove (3) we will use the arithmetic-geometric mean inequality ($A\geq G$) for three positive numbers. Because $xyz=1$, we have:

$$\frac{1}{3}x^2z+\frac{1}{3}yz^2+\frac{1}{3}yz^2\geq 3.\ \sqrt[3]{\frac{1}{3^3}\cdot(xyz)^2\cdot z^3}=z,$$

$$\frac{1}{3}x^2z+\frac{1}{3}x^2z+\frac{1}{3}xy^2 \ge 3.\ \sqrt[3]{\frac{1}{3^3}\cdot(xyz)^2\cdot x^3} = x,$$
$$\frac{1}{3}xy^2+\frac{1}{3}xy^2+\frac{1}{3}yz^2 \ge 3.\ \sqrt[3]{\frac{1}{3^3}\cdot(xyz)^2\cdot y^3} = y.$$

Now add the above three inequalities and obtain (3).

The inequality (4) could be proved in a similar manner. Namely, by $A \ge G$ and $xyz = 1$ we get:

$$\frac{1}{3}x^2y+\frac{1}{3}x^2y+\frac{1}{3}y^2z \ge 3.\ \sqrt[3]{\frac{1}{3^3}\cdot xyz\cdot x^3y^3} = xy,$$

$$\frac{1}{3}y^2z+\frac{1}{3}y^2z+\frac{1}{3}xz^2 \ge 3.\ \sqrt[3]{\frac{1}{3^3}\cdot xyz\cdot y^3z^3} = zy,$$

$$\frac{1}{3}x^2y+\frac{1}{3}xz^2+\frac{1}{3}xz^2 \ge 3.\ \sqrt[3]{\frac{1}{3^3}\cdot xyz\cdot x^3z^3} = xz.$$

Again add the last three inequalities and obtain (4).

Now (2) is a consequence of (3) and (4) by adding them. The equality in (1) holds true if and only if $x=y=z=1$.

Proof 2: The inequality (1) is equivalent to the following one (adding 3 to both sides):

$$\frac{x+y}{1+xy}+1+\frac{y+z}{1+yz}+1+\frac{z+x}{1+zx}+1 \ge 6,$$

and respectively to

$$\frac{(x+1)(y+1)}{1+xy}+\frac{(y+1)(z+1)}{1+yz}+\frac{(z+1)(x+1)}{1+zx} \ge 6.$$

Since $xyz=1$, we have from here that $xy=\frac{1}{z},\ yz=\frac{1}{x},\ zx=\frac{1}{y}$ and consequently:

$$\frac{z(x+1)(y+1)}{z+1}+\frac{x(y+1)(z+1)}{x+1}+\frac{y(z+1)(x+1)}{y+1} \ge 6. \qquad (5)$$

Now by $A \geq G$ we get:

$$\frac{z(x+1)(y+1)}{z+1}+\frac{x(y+1)(z+1)}{x+1}+\frac{y(z+1)(x+1)}{y+1}\geq$$

$$\geq 3\sqrt[3]{xyz(x+1)(y+1)(z+1)}=3.\ \sqrt[3]{(x+1)(y+1)(z+1)}$$

$$\geq 3\sqrt[3]{2\sqrt{x}\cdot 2\sqrt{y}\cdot 2\sqrt{z}}=3.\ \sqrt[3]{2^3\sqrt{xyz}}=6.$$

This ends the proof.

Proof 3: We will use the substitution $x=\frac{a}{b},\ y=\frac{b}{c},\ z=\frac{c}{a};\ (a,b,c>0)$. It implies that the given inequality is equivalent to:

$$\frac{ac+b^2}{b(a+c)}+\frac{ab+c^2}{c(a+b)}+\frac{bc+a^2}{a(b+c)}\geq 3$$

$$\Leftrightarrow\quad \frac{ac+b^2}{b(a+c)}+1+\frac{ab+c^2}{c(a+b)}+1+\frac{bc+a^2}{a(b+c)}+1\geq 6$$

$$\Leftrightarrow\quad \frac{(b+c)(a+b)}{b(a+c)}+\frac{(b+c)(a+c)}{c(a+b)}+\frac{(a+b)(a+c)}{a(b+c)}\geq 6\,. \tag{6}$$

Apply again the $A \geq G$ inequality:

$$\frac{(b+c)(a+b)}{b(a+c)}+\frac{(b+c)(a+c)}{c(a+b)}+\frac{(a+b)(a+c)}{a(b+c)}\geq$$

$$\geq\ 3.\ \sqrt[3]{\frac{(a+b)(b+c)(a+c)}{abc}}\geq 3\sqrt[3]{\frac{2\sqrt{ab}\cdot 2\sqrt{bc}\cdot 2\sqrt{ac}}{abc}}=$$

$$=3.\ \sqrt[3]{\frac{2^3\sqrt{a^2b^2c^2}}{abc}}=6,\ \text{q.e.d.}$$

Thus, (6) is proved and this implies the validity of (1). The equality holds true if and only if $a = b = c$, i.e. $x = y = z = 1$.

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НЯКОЛКО ДОКАЗАТЕЛСТВА НА ЕДНО АЛГЕБРИЧНО НЕРАВЕНСТВО

Шефкет Арсланагич

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