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RECONSTRUCTING THE MATHEMATICS CURRICULUM FOR POST-16 STUDENTS: A COMPARISON OF THE APPROACHES IN ENGLAND AND BULGARIA

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Abstract. A major reform of the mathematics curriculum was published in England in 2016, implemented in 2017 and examined in 2019. In Bulgaria, the analogous reform was published in 2018, implemented in 2020 and examined in 2022. This paper compares the curricula at four levels: general and specific objectives, overarching themes, topics covered, and detailed content statements for each topic. The paper concludes that the real differences between the teaching of mathematics in England and Bulgaria lie not in the mathematics itself but in the overall curriculum and assessment frameworks within which mathematics is taught.

Keywords: mathematics curriculum; reforms; assessment frameworks

1. The Context

Since 2016, both Bulgaria and England have undertaken major reforms of their mathematics curricula. In England, a new mathematics curriculum was introduced in 2017 and examined for the first time in 2019; while in Bulgaria, the new mathematics curriculum was introduced in 2020 and examined for the first time in 2022. The purpose of this paper is to examine in greater depth the similarities and differences in the mathematics curricula in England and Bulgaria. A comparison of the assessment procedures will be addressed in a future paper that is in preparation.

2. Methodology

The paper is based on a comparative analysis of curriculum documents published by the Department for Education in England and the Ministry of Education and Science in Bulgaria. The documents were produced in very

different formats, so the first task was to construct a common framework that would enable them to be interpreted and compared:

- General and Specific Objectives;
- Overarching Themes;
- Topics; and
- Detailed Content Statements.

3. The Mathematics Curriculum

The education of older students is constrained by external qualifications which requires a detailed specification of the content of all subjects examined in schools. At the highest level of generality, the Department for Education in England and the Ministry of Education and Science in Bulgaria take similar approaches to the mathematics curriculum. They specify, at the highest level, General and Specific Objectives; followed by Overarching Themes; then the Topics covered and Detailed Content Statements for each topic.

3.1. General and Specific Objectives

The highest-level statements about the mathematics curriculum are entitled "General and Specific Objectives" in England, but "Aims of Mathematics Education" in Bulgaria.

Table 1 indicates that only General Objectives are published by the Department of Education in England, for example: Using mathematical skills and techniques to solve challenging problems which require students to decide on the solution strategy; and Representing situations mathematically and understanding the relationship between problems in context and mathematical models that may be applied to solve them.

The list of objectives published by the Ministry of Education and Science in Bulgaria, however, includes General Objectives, for example: Deepening logical knowledge and skills, forming a logical culture and learning mathematical language; and Specific Objectives, for example: Acquiring knowledge of the concept of polynomials, of some types of polynomials and their elements, knowledge about the faces, surfaces and volumes of studied polynomials and developing skills for their application.

Table 1. The General and Specific Objectives of the Mathematics Curriculum in England and Bulgaria

ENGLAND	**BULGARIA**
Specification for mathematics	**Mathematics curriculum for grade XI and XII**
Aims and objectives	**Objectives of mathematics education**
Understanding mathematics and mathematical processes in a way that promotes confidence, fosters enjoyment and provides a strong foundation for progress to further study. Extending the range of mathematical skills and techniques. Understanding coherence and progression in mathematics and how different areas of mathematics are connected. Applying mathematics in other fields of study and be aware of the relevance of mathematics to the world of work and to situations in society in general. Using mathematical knowledge to make logical and reasoned decisions in solving problems both within pure mathematics and in a variety of contexts, and communicate the mathematical rationale for these decisions clearly. Reasoning logically and recognise incorrect reasoning. Generalising mathematically. Constructing mathematical proofs. Using mathematical skills and techniques to solve challenging problems which require students to decide on the solution strategy. Recognising when mathematics can be used to analyse and solve a problem in context. Representing situations mathematically and understanding the relationship between problems in context and	Deepening of logical knowledge and skills, formation of logical culture and mastering of mathematical language. Mastering the major applications by extending intragranular and intersubject links. Mastering scientific methods and ideas. Establishing such relationships, between the teacher and students, between the students themselves and between the students and the community, enabling the expression of the personal qualities of each student and the formation of civic position: Bringing mathematics education up to European standards while maintaining national traditions. Building habits for protecting the environment and your own health.
	Grade XI (General Mathematics)
	Knowledge of some specific numerical series and their applications. Expanding knowledge of trigonometric functions and their applications. Laying the foundations of statistical knowledge through familiarity with the mechanisms for presenting data.
	Grade XI (Profile Mathematics)
	Mastering the properties of logarithms, exponential and logarithmic functions and their applications. Mastering the interrelationships of points, straight lines and planes in space, relations of "parallel" and "perpendicularity" and building skills to prove the claims related to them.

ENGLAND	BULGARIA
mathematical models that may be applied to solve them. Drawing diagrams and sketch graphs to help explore mathematical situations and interpret solutions. Making deductions and inferences and draw conclusions by using mathematical reasoning. Interpreting solutions and communicate their interpretation effectively in the context of the problem. Reading and comprehending mathematical arguments, including justifications of methods and formulae, and communicating their understanding. Reading and comprehending articles concerning applications of mathematics and communicating their understanding. Using technology such as calculators and computers effectively and recognising when such use may be inappropriate. Taking increasing responsibility for their own learning and the evaluation of their own mathematical development.	Knowledge of the concept of polynomials, some types of polyhedra and their elements, knowledge of faces on their surfaces and volumes, and of skills in their application. Expanding knowledge of vectors in space and their applications. **Grade XII (General Mathematics)** Acquiring knowledge related to the mutual positions of points, straight lines and planes in space, relations "parallel", "perpendicularity" and the image "orthogonal design". Acquiring knowledge about the concept of polyhedra, about some types of polyhedra and their elements, knowledge about the faces of surfaces and volumes of studied polynomials and forming skills for their application. Summarizing the knowledge of the studied numerical sets. Summarizing knowledge of the studied features, their properties and applications. Expanding knowledge of equations and inequalities. **Grade XII (Profile Mathematics)** Acquiring knowledge about certain types of rotary bodies, their elements, faces on surfaces and volumes and forming skills for their application. Mastering the elements of mathematical analysis and their applications. Assimilation of elements of analytical geometry in the plane and their applications. Upgrading knowledge by probability and statistics. Introducing the set of complex numbers and forming the skills to perform operations with complex numbers.

3.2. Overarching Themes

Table 2 indicates that, although the terminology differs, both the Department for Education and the Ministry of Education and Science have adopted similar approaches when giving a high-level overview of the mathematics curriculum.

Table 2. A High-level View of Mathematics in England and Bulgaria

ENGLAND	**BULGARIA**
Overarching Themes	
Argument, language and proof	Knowledge, skills and attitudes
Problem solving	Areas of competence
Modelling	Establishing cross-domain links
The use of technology	Practical activities
The use of data in statistics	Use of information technology

3.3. Topics

The next level of analysis, at the level of Topics, is shown in Table 3. It is at this level that some of the differences in approach become more apparent. This suggests that General Mathematics in Bulgaria and Mathematics in England are not directly equivalent. In England, there is no equivalent to General Mathematics in Bulgaria. The only students aged 16 and over who study mathematics in England are those who have chosen to study mathematics to university entrance level. So, is mathematics in England equivalent to Profile Mathematics in Bulgaria?

Figures 1, 2 and 3 have been prepared to further this analysis. Figure 1 illustrates very clearly the lack of equivalence between General Mathematics in Bulgaria and Mathematics in England. With the exception that students in Bulgaria study Geometry and Inequalities, the mathematics curriculum in England is much wider. So, is Mathematics equivalent to Profile Mathematics? This is addressed in Figure 2, which indicates that Mathematics in England and Profile Mathematics in Bulgaria map more closely onto one another. Some topics are included in Profile Mathematics in Bulgaria but not in Mathematics in England, for example, 3-Dimensional Geometry and Geometric Modelling. Conversely, other topics are included in Mathematics in England but not in Profile Mathematics in Bulgaria, for

example, Integration, Quantities and Units in Mechanics, Kinematics and Forces and Newton's Laws.

Table 3. Topics Covered in Mathematics in England and Bulgaria

ENGLAND	**BULGARIA**
Topics	**Themes**
Mathematics	**General Mathematics**
Mathematics	**Grade XI**
Proof	Exponentials and logarithm
Algebra and Functions	2-Dimensional Geometry
Co-ordinate Geometry in the (x,y) plane	Trigonometry
Sequences and Series	Probability
Trigonometry	**Grade XII**
Exponential and Logarithms	Statistics
Differentiation	Equations
Integration	Inequalities
Numerical Methods	Extension Activities
Vectors	
Statistics	**Profile Mathematics**
Statistical Sampling	**Grade XI**
Data Presentation and Interpretation	**Geometry**
Probability	Vectors and coordinates
Statistical Distributions	Co-ordinate Geometry in the (x, y) plane
Statistical Hypothesis Testing	3-Dimensional Geometry
Mechanics	**Mathematical Analysis**
Quantities and Units in Mechanics	Polynomials of a variable
Kinematics	Numbers
Forces and Newton's Laws	Functions Series and Differentiation
	Grade XII
Further Mathematics	**Practical Mathematics**
Proof	Applications of Mathematical Analysis
Complex numbers	Geometric models
Matrices	Empirical distributions
Further Algebra and Functions	**Probabilities and data analysis**
Further Calculus	Probability
Further Vectors	Random value
Polar Coordinates	Binomial distribution
Hyperbolic Functions	Normal distribution
Differential Equations	

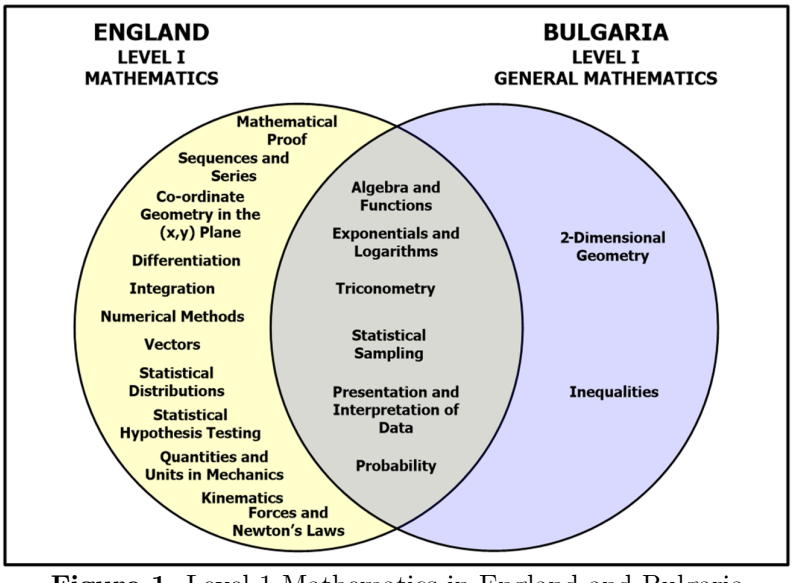


Figure 1. Level 1 Mathematics in England and Bulgaria

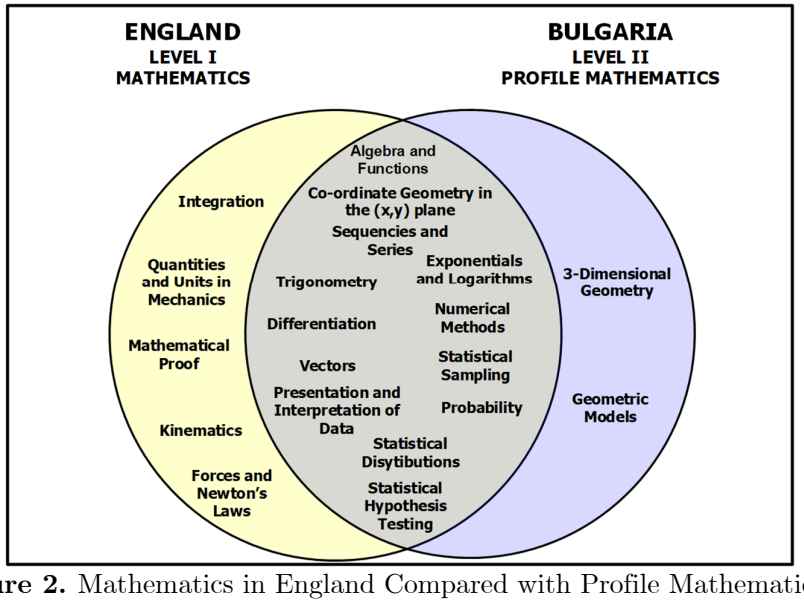


Figure 2. Mathematics in England Compared with Profile Mathematics in Bulgaria

At the level of topics, therefore, as defined by the Department for Education and the Ministry of Education and Science, there is a major overlap between Mathematics in England and Profile Mathematics in Bulgaria. Where, then, does Further Mathematics fit? This is addressed in Figure 3.

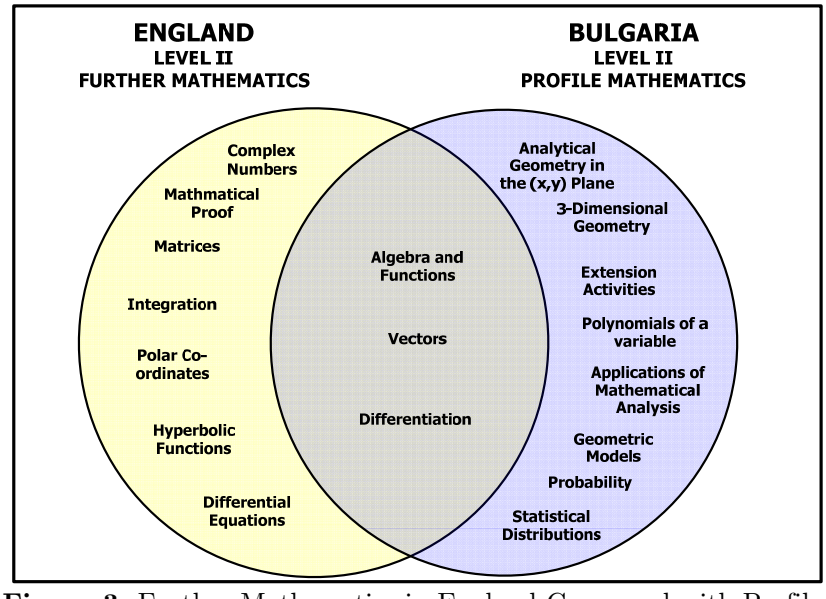


Figure 3. Further Mathematics in England Compared with Profile Mathematics in Bulgaria

Because the majority of topics covered in Further Mathematics are clearly distinct from those covered in Mathematics, Figure 3 indicates that they are also distinct from the topics in Profile Mathematics. Therefore, with the exception of Differentiation, Algebra and Functions, and Vectors; the topics covered in Further Mathematics are clearly distinct from those covered in Profile Mathematics, for example, integration, Complex Numbers, Polar Co-ordinates, Hyperbolic Functions and Differential Equations. This raises the question of where Further Mathematics fits. Because of the differences between the curriculum and the assessment frameworks in England and Bulgaria, there is no direct equivalence.

3.4. Detailed Content Statements

Tables 4a and 4b present the detailed content statements for "Exponentials and Logarithms". Tables 5a and 5b present the detailed content statements for "Vectors".

Table 4a. A Detailed Statement of the Content for Exponentials and Logarithms in England

ENGLAND
Detailed Content Statements for Exponentials and Logarithms
• Know and use the function a^x and its graph, where a is positive • Know and use the function e^x and its graph • Know that the gradient of$e^{kx}$is equal to $ke^{kx}$and hence understand why the exponential model is suitable in many applications • Know and use the definition of $log_a x$ as the inverse of a^x , where a is positive and $x \geq 1$ • Know and use the function and its graph • Know and use $ln\, x$as the inverse function of e^x • Understand and use the laws of logarithms: • $log_a x + log_a y = log_a(xy)$; $log_a x - log_a y = \left(\frac{x}{y}\right)$; $k\, log_a x = log_a x^k$ • (including, for example, $k = -1$and $k = -\frac{1}{2}$) • Solve equations of the form $a^x = b$ • Use logarithmic graphs to estimate parameters in relationships of the form $y = ax^n$ and $y = kb^x$, given data for x and y • Understand and use exponential growth and decay; use in modelling (examples may include the use of $e$in continuous compound interest, radioactive decay, drug concentration decay, exponential growth as a model for population growth); consideration of limitations and refinements of exponential models

Table 4b. A Detailed Statement of the Content for Exponentials and Logarithms in Bulgaria

BULGARIA
Numbers and Algebra
Knowledge, skills and attitudes As a result of their education, the student: • knows the concept of the nth root and its properties; • knows the concepts of exponential and logarithm and their basic properties; performs identical transformations of irrational expressions containing square and cubic roots and 4th root.

Topics	**Expected Competencies at the end of the topic**	**New concepts**
1. Exponentials and Logarithms 1.1. Properties of the 3rd root. 1.2. Properties of the nth root. 1.3. Transformation of irrational expressions. 1.4. Function graphs $y = \sqrt{x}$, $y = x^3$ и $y = \sqrt[3]{x}$ 1.5. Exponential with rational exponent properties. 1.6. Convert expressions containing an exponential with a rational exponent. 1.7. Exponential function. Graphics. 1.8. Logarithm. Basic properties. Comparison of logarithms. Graph of logarithmic function. 1.9. Logarithm of a product and quotient, exponential and root.	• knows the concept of nth root and its properties; • be able to translate irrational expressions, containing square and cubic roots, and root 4-you; • knows the concept of degree with a rational indicator and its properties; • able to translate expressions containing degrees with rational indicator; • knows the concept of logarithm and its properties; • be able to apply logarithm properties to expression transformation; • be able to find the elements of a logarithm value, basis or argument in the presence of the other two dimensions; • be able to recognize the power graphs, indicative and logarithmic function; • read and interpret information provided by graphics; • can solve practical problems with a calculator.	Third root (cubic root), nth root, logarithm, basis, logarithm, anti-logarithm.

Table 5a. A Detailed Statement of the Content for Vectors in England

ENGLAND
Detailed Content Statement of Vectors
Mathematics
• Use vectors in two and in three dimensions • Calculate the magnitude and direction of a vector and convert between component form and magnitude/direction form • Add vectors diagrammatically and perform the algebraic operations of vector addition and multiplication by scalars, and understand their geometrical interpretations • Understand and use position vectors; calculate the distance between two points represented by position vectors

• Use vectors to solve problems in pure mathematics and in context, including forces and kinematics
Further Mathematics
• Understand and use the vector and Cartesian forms of an equation of a straight line in 3D • Understand and use the vector and Cartesian forms of the equation of a plane • Calculate the scalar product and use it to express the equation of a plane, and to calculate the angle between two lines, the angle between two planes and the angle between a line and a plane • Check whether vectors are perpendicular by using the scalar product • Find the intersection of a line and a plane • Calculate the perpendicular distance between two lines, from a point to a line and from a point to a plane

Table 5b. A Detailed Statement of the Content for Vectors in Bulgaria

BULGARIA		
Geometry		
Knowledge, skills and attitudes Vectors are the basis of different parts of physics, especially in mechanics when studying the movement of bodies and the forces that act on them in different situations.		
Topics	**Expected Competencies at the end of the topic**	**New concepts**
Vectors	The student:	
1.1. Linear dependence and independence of vectors in the plane and in space.	• knows the concepts of linear dependence and independence of vectors; • is able to determine linear dependence and independence of vectors.	Linear dependence and of vectors, linear independence.
1.2. Vector base in the plane and in the space.	• knows that if it is in the plane (or space) selected vector base, each vector in the plane (or space) can be represented as linear combination of base vectors; • is able to decompose a vector as linear combination of vectors from base to specific situations in the plane and in space.	Vector base.
1.3. A scalar product of two vectors.	• knows the scalar product in two vectors; • is able to find a scalar product of two vectors;	Scalar product

	• knows the properties of a scalar product.	
1.4. Scalar application product - vector length; angle between two vectors.	• is able to find the length of a vector; • is able to find an angle between two vectors.	Angle between two vectors.
1.5. Vector coordinates in the plane. Rectangular coordinate system.	• knows the concept of vector coordinates in plane rectangular coordinate system; • is able to find the coordinates of a point; • is able to find the distance between two points.	Vector coordinates.
1.6. Vector coordinates in the plane. Rectangular coordinate system.	• is able to find the sum of vectors set in coordinate form; • is able to find the product of a number with a vector, set in coordinate form; • is able to determine the coordinates of a vector, set by a linear combination of other vectors; • is able to find the scalar product of two the vector in the plane by their coordinates; • is able to find an angle between two vectors, set by their coordinates.	Sum of vectors given by coordinates, linear combination of vectors, a scalar product in two vectors, given coordinates, angle between two vectors given by coordinates.

Despite differences in presentation style, the examples analysed suggest a high degree of similarity in the selected shared topics. However, this is not peculiar to "Exponentials and Logarithms" and "Vectors". These were selected as examples; the same pattern appears across the topics taught in both England and Bulgaria. So, what does this say?

Primarily, the differences in mathematics teaching between England and Bulgaria reflect differences in their overall curricula and educational frameworks rather than differences in the mathematics itself.

This is perhaps unsurprising: mathematics is mathematics. However, if Ellenberg (see Postscript and Paradox, below) had had greater influence and school mathematics were "Simple and Profound" rather than being "Complex but Shallow" students in upper secondary schools might engage

with topics such as Benacerref's Dilemma (Benacerref, 1973) "what is necessary for mathematical truth makes mathematical knowledge impossible"; Pluralism and the foundations of mathematics (Hellman & Bell, 2006); or Pluralism: Beyond the One and Only Truth (Horgan, 2019).

In that case, the content statements might well be very different – and legitimately so.

4. Summary

At the level of individual topics covered in both Mathematics (Level II in England) and Profile Mathematics (Level II in Bulgaria) there is substantial alignment between the Detailed Content Statements.

At the next level, the range of topics covered also shows a similar pattern. Of course, some Topics are included in Profile Mathematics in Bulgaria that are not included in Mathematics in England, for example, 3-Dimensional Geometry and Geometric Modelling; others are included in Mathematics in England but not in Profile Mathematics in Bulgaria, for example, integration, Complex Numbers, Polar Co-ordinates, Hyperbolic Functions and Differential Equations.

The overall range of topics is, however, similar. At the highest level of generality, there is a similar approach to the mathematics curriculum. It is specified in terms of General and Specific Objectives and Overarching Themes and these are remarkably similar.

The real differences between the teaching of mathematics in England and Bulgaria lie not in the mathematics itself but in the overall curriculum and assessment frameworks within which Mathematics is taught. Students in England can choose the three or four subjects that they study after the age of 16 – there is no General Mathematics programme. Students in Bulgaria elect or are selected to follow profiles (pathways) at the age of 14 within which they have limited freedom of choice thereafter.

5. Paradox and PostScript

Despite reforms in both countries, little has been done to address the challenge issued by Jordan Ellenberg (Ellenberg, 2014) in "How not to be wrong. The hidden maths of everyday life".

Table 6. Ellenberg's Quadrant

	Jordan Ellenberg's Quadrants	
Profound	This should be the focus for school/college mathematics ← ↑	The Riemann Hypothesis, Fermat's Last Theorem, The Poincaré Conjecture, P vs. NP, Gödel's Theorem
Shallow	$1 + 2 = 3$ $\sin(2x) = 2\sin x \times \cos x$	Much of school/college mathematics $\int x^{-1}\sqrt{x^3 - 1}.dx =$ $\sqrt[\frac{2}{3}]{x^3 - 1}.\left(1 - \frac{\tanh^{-1}\left(\sqrt{1 - x^3}\right)}{\sqrt{1 - x^3}}\right)$
	Simple	**Complicated**

5.1. Simple but shallow

Basic arithmetic facts, like $1 + 2 = 3$, are simple and shallow. So are basic identities like $sin(2x) = 2\,sin\,x.cos\,x$ or the quadratic formula: they might be slightly harder to convince yourself of than $1 + 2 = 3$, but in the end, they don't have much conceptual weight.

5.2. Complicated but Shallow – Much of school/college mathematics

You have the problem of multiplying two ten-digit numbers, or the computation of an intricate definite integral. It is conceivable that, for some reason, you might need to know the answer to such a problem, and it's undeniable that it would be somewhere between annoying and impossible to work it out by hand; or, it might take some serious schooling even to understand what's being asked for. However, knowing those answers does not necessarily enrich your knowledge about the world.

5.3. Simple and Profound – This should be the focus for school/college mathematics

Mathematical ideas that can be engaged with directly and profitably, whether your mathematical training stops at pre-algebra or extends much

further. And they are not "mere facts," like a simple statement of arithmetic— they are principles, whose application extends far beyond the things you're used to thinking of as mathematical. They are the go-to tools on the utility belt, and used properly they can help you avoid errors.

5.4. Complicated and Profound

This is where professional mathematicians aim to spend much of their time. It's where the celebrity theorems and conjectures live:

- The Riemann Hypothesis,
- Fermat's Last Theorem,
- The Poincaré Conjecture,
- P vs. NP,
- Gödel's Theorem ...

Each one of these theorems involves ideas of deep meaning, fundamental importance, mind-blowing beauty, and brutal technicality.

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