

PROBLEM 6. FROM IMO'2018

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Abstract. The International Mathematical Olympiad is one of the respectable events and one of the most long-lived international educational and scientific competitions. It is the largest, oldest and most prestigious scientific Olympiad for high school students. The 59th edition of the event took place in Cluj-Napoca, Romania, 3 – 14 July 2018. The present paper is dedicated to the sixth problem on the Olympiad paper. A detailed analysis of the problem is proposed in a methodological way, which will be useful for students and teachers in the preparatory process for future participations in mathematical competitions.

Keywords: Olympiad; problem solving

The problem 6 on the paper of the 59th International mathematical Olympiad was solved fully (7 points) by 18 participants, 5 students were marked with 6 points, 2 with 5 points, 5 with 4 points, 11 with 3 points, 26 with 2 points, 108 with 1 point and 419 with 0 points. The mean result of all the 594 participants in the Olympiad from 107 countries is 0. 638, which shows that the problem is hard and needs a detailed analysis.

Problem 6. A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$. Prove that $\angle BXA + \angle DXC = 180^\circ$.

Lemma 1. Each convex quadrilateral has a unique interior point X such that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$.

Proof: The following cases are possible:

1) If $ABCD$ is a parallelogram, then the point X is the intersection point of the diagonals. This follows from the equality of the cross-opposite angles of the parallelogram. Reversely, the equality of the mentioned angles implies that the point X should belong to each of the diagonals and consequently it belongs to both the diagonals simultaneously. Thus the point X is unique.

2.1) If $ABCD$ is a trapezoid with $AB \parallel CD$ and $BC \cap AD = W$, then the common point of the diagonal AC and the circumcircle $k(BDW)$ of $\triangle BDW$ is the desired point X . The proof is the following:

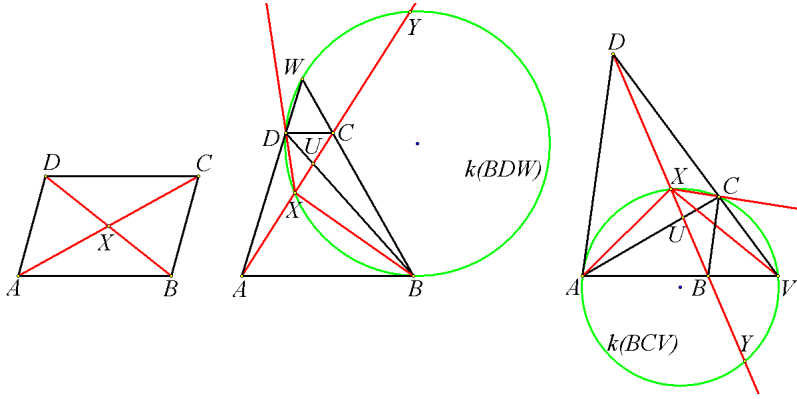
$$\begin{aligned} \text{i) } \angle XAB &= \angle XCD \text{ (cross-opposite angles);} \\ \text{ii) } \angle XBC &= \frac{\widehat{XDW}}{2} = \frac{\widehat{XD} + \widehat{DW}}{2} = \angle XDA. \end{aligned}$$

Reversely, the equality $\angle XAB = \angle XCD$ implies, that the point X belongs to the diagonal AC , while the equality $\angle XBC = \angle XDA$ is verified when X is on the circle $k(BDW)$. Since AC and $k(BDW)$ have only one common point, which is in the interior of $ABCD$, then the point X is unique. The second intersection point Y of the line AC and $k(BDW)$ is such, that $\angle YAB = 180^\circ - \angle YCD$ and $\angle YBC = 180^\circ - \angle YDA$.

2.2) If $ABCD$ is a trapezoid with $BC \parallel AD$ and $AB \cap CD = V$, then the common point of the diagonal BD and the circumcircle $k(ACV)$ of $\triangle ACV$ is the desired point X . The proof is the following:

$$\begin{aligned} \text{i) } \angle XAB &= \frac{\widehat{XCV}}{2} = \frac{\widehat{XC} + \widehat{CV}}{2} = \angle XCD; \\ \text{ii) } \angle XBC &= \angle XDA \text{ (cross-opposite angles).} \end{aligned}$$

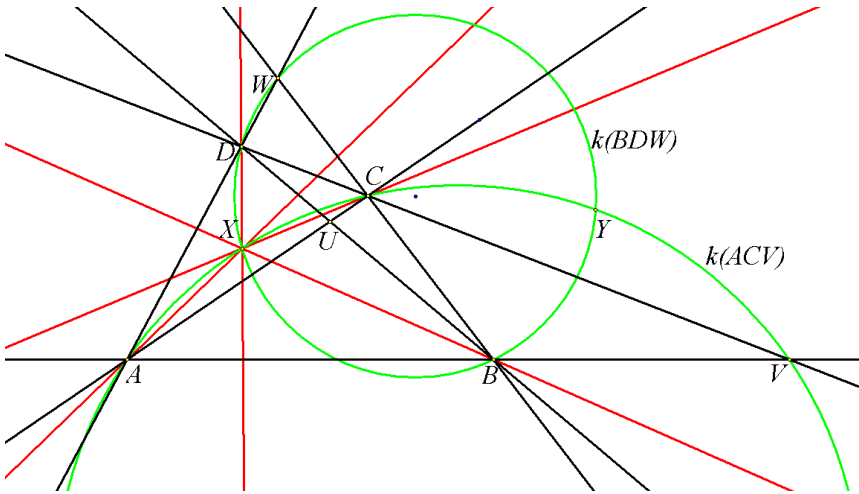
The uniqueness of the point X could be proved in the same manner as in the previous case. The second common point Y of the line BD and $k(ACV)$ is such, that $\angle YAB = 180^\circ - \angle YCD$ and $\angle YBC = 180^\circ - \angle YDA$.



3) If $ABCD$ is without parallel sides and $AB \cap CD = V$, $BC \cap AD = W$, then the interior point of $ABCD$, which is the intersection point of the circumcircles $k(ACV)$ and $k(BDW)$ of $\triangle ACV$ and $\triangle BDW$, respectively, is the desired point X . The proof is the following:

$$\text{i) } \angle XAB = \frac{\widehat{XCV}}{2} = \frac{\widehat{XC} + \widehat{CV}}{2} = \angle XCD;$$

$$\text{ii) } \angle XBC = \frac{\widehat{XDW}}{2} = \frac{\widehat{XD} + \widehat{DW}}{2} = \angle XDA.$$



Reversely, the equality $\angle XAB = \angle XCD$ means, that the point X belongs to the circle $k(ACV)$, while the equality $\angle XBC = \angle XDA$ is verified when X is on the circle $k(BDW)$. Since AC and $k(BDW)$ have only one common point, which is in the interior of $ABCD$, then the point X is unique. The second intersection point Y of $k(ACV)$ and $k(BDW)$ is such, that $\angle YAB = 180^\circ - \angle YCD$ and $\angle YBC = 180^\circ - \angle YDA$.

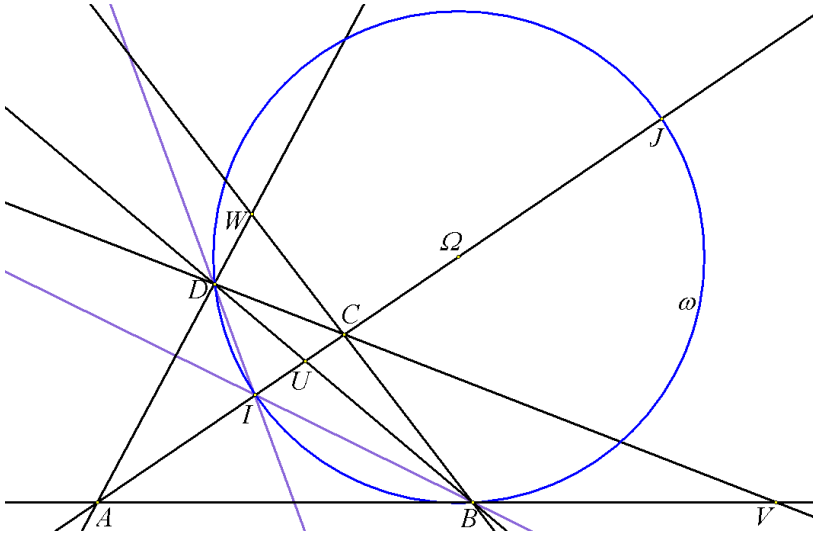
Lemma 2. If the sides of the convex quadrilateral $ABCD$ satisfy the equality $AB \cdot CD = BC \cdot DA$, then:

a) the angular bisectors of the angles ABC and CDA meet in the point I from the diagonal AC ;

b) the angular bisectors of the angles BAD and DCB meet in the point I' from the diagonal BD .

Proof. Rewrite the given equality in the form $\frac{BA}{BC} = \frac{DA}{DC}$. If I is such a point on the diagonal AC , that $\frac{IA}{IC} = \frac{BA}{BC}$, then BI is the angular bisector of $\angle ABC$.

On the other hand, the equality $\frac{IA}{IC} = \frac{DA}{DC}$ is true, which means, that DI is the angular bisector of $\angle CDA$. The assertion b) could be obtained analogously.

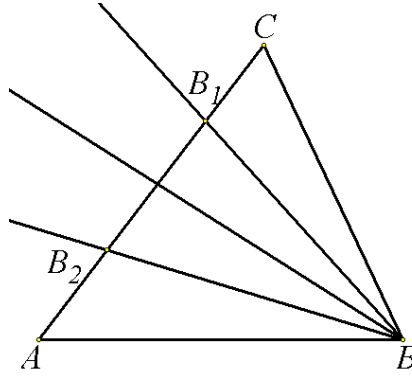


In fact, if $BA \neq BC$, then there exists a point J on the line BC with the property $\frac{JA}{JC} = \frac{BA}{BC}$. The circle ω with diameter IJ is the locus of the points M , for which $\frac{MA}{MC} = \frac{BA}{BC}$. The circle ω is known to be the *Apollonius circle* for the segment AC under the ratio $\frac{BA}{BC}$. For this reason, if an arbitrary $\triangle ABC$ is given, then the point D from the Apollonius circle ω is the fourth vertex of the quadrilateral $ABCD$, for which $AB \cdot CD = BC \cdot DA$. In such a way we come upon an idea for the construction of a quadrilateral satisfying the conditions of the problem under consideration. On the other hand, lemma 1 gives a possibility to construct the point X for this quadrilateral.

Lemma 3. Let ABC be an arbitrary triangle, while B_1 and B_2 be such points on the line AC , that the lines BB_1 and BB_2 are symmetric with respect to the angular bisector of $\angle ABC$. Then $\frac{\overline{CB_1}}{\overline{AB_1}} \cdot \frac{\overline{CB_2}}{\overline{AB_2}} = \frac{BC^2}{AB^2}$.

Proof: For the areas of the triangles BCB_1 and BAB_2 we have, that $\frac{S_{BCB_1}}{S_{BAB_2}} = \frac{CB_1}{AB_2}$ and $\frac{S_{BCB_1}}{S_{BAB_2}} = \frac{BC \cdot BB_1 \cdot \sin \angle B_1BC}{AB \cdot BB_2 \cdot \sin \angle B_2BA} = \frac{BC \cdot BB_1}{AB \cdot BB_2}$. Consequently $\frac{CB_1}{AB_2} = \frac{BC \cdot BB_1}{AB \cdot BB_2}$.

Analogously, using the areas of the triangles CBB_2 and ABB_1 we get the equality $\frac{CB_2}{AB_1} = \frac{BC \cdot BB_2}{AB \cdot BB_1}$. Multiplying the obtained equalities we find, that $\frac{CB_1}{AB_1} \cdot \frac{CB_2}{AB_2} = \frac{BC^2}{AB^2}$.



Lemma 4. If the convex quadrilateral $ABCD$ satisfies the equality $AB \cdot CD = BC \cdot DA$, then:

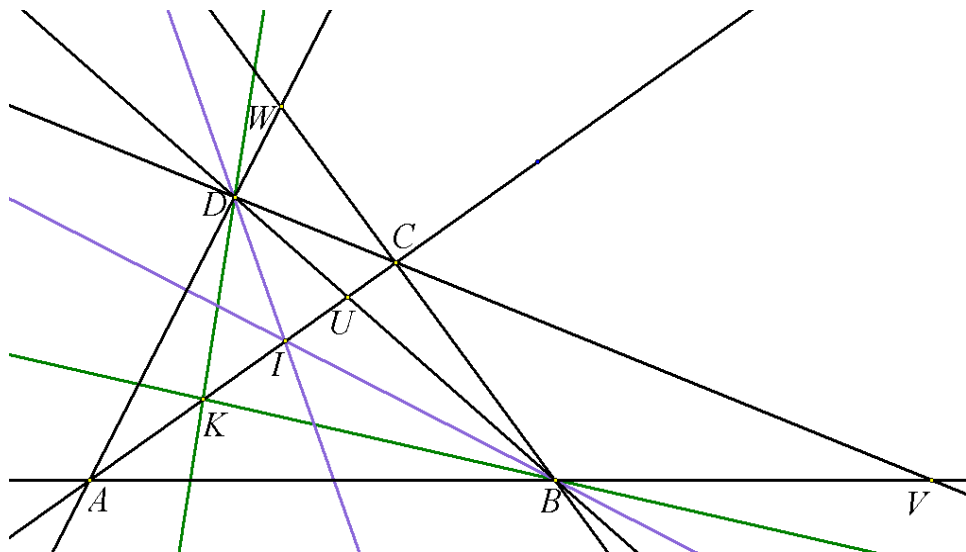
- a) the symmetric images of the diagonal BD with respect to the angular bisectors of $\angle ABC$ and $\angle CDA$ meet in a point K from the diagonal AC ;
- b) the symmetric images of the diagonal AC with respect to the angular bisectors of $\angle BAD$ and $\angle DCB$ meet in a point K' from the diagonal BD .

Proof. Denote by U the common point of the diagonals AC and BD . Let the symmetric images of BD with respect to the angular bisectors of $\angle ABC$ and $\angle CDA$ meet AC in the points K_1 and K_2 , respectively. Applying lemma 3 to $\triangle ABC$ we obtain the equality $\frac{CU}{AU} \cdot \frac{CK_1}{AK_1} = \frac{BC^2}{AB^2}$. Applying lemma 3 to $\triangle CDA$

we obtain $\frac{AU}{CU} \cdot \frac{AK_2}{CK_2} = \frac{DA^2}{CD^2}$. Multiplying the two equalities we deduce, that

$$\frac{CK_1 \cdot AK_2}{AK_1 \cdot CK_2} = \left(\frac{BC \cdot DA}{AB \cdot CD} \right)^2 = 1. \text{ Therefore, } \frac{AK_2}{CK_2} = \frac{AK_1}{CK_1}. \text{ Since the points } K_1 \text{ and } K_2$$

K_2 belong to the segment AC , the last equality means, that the points K_1 and K_2 coincide. The assertion for the point K' could be proven analogously.



It follows from lemma 4 that the point I is the center of the incircle of $\triangle BDK$. Thus, AC is the angular bisector of $\angle BKD$. Consequently $180^\circ = \angle AKD + \angle DKC = \angle AKD + \angle BKC$. Thus we obtain the following

Conclusion. The equalities $\angle AKD + \angle BKC = 180^\circ$ and $\angle BK'A + \angle CK'D = 180^\circ$ are satisfied.

Lemma 5. If the convex quadrilateral $ABCD$ satisfies the equality $AB \cdot CD = BC \cdot DA$, then:

a) The second common point X of the circumcircles $k(BCK)$ and $k(DAK)$ of $\triangle BCK$ and $\triangle DAK$, respectively, is such that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$.

b) The second common point X of the circumcircles $k(BCK')$ and $k(DAK')$ of $\triangle BCK'$ and $\triangle DAK'$, respectively, is such that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$.

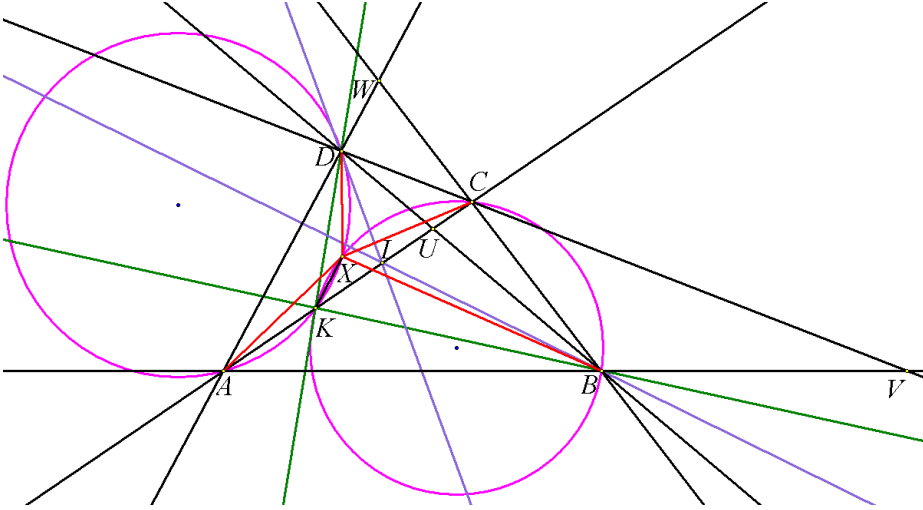
Proof. We will consider the first case only. Let $AB \cap AD = V$ and the second common point of $k(BCK)$ and $k(DAK)$ be X . In order to prove the first equality it is enough to establish, that the point X belongs to the circumcircle $k(ACV)$ of $\triangle ACV$ (as shown in lemma 1). The following equalities are true:

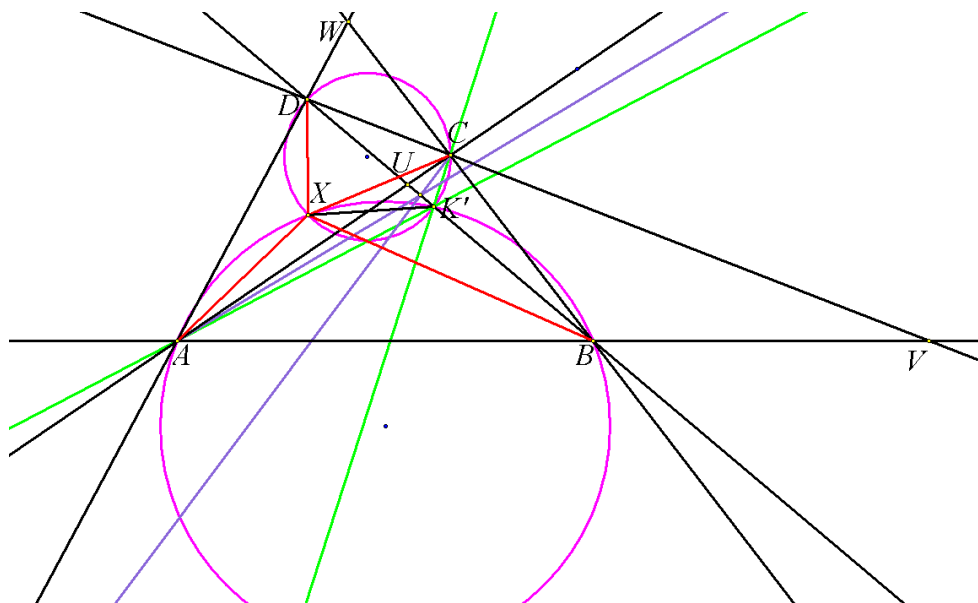
$$\begin{aligned}\angle CXA &= \angle CXK + \angle KXA = 180^\circ - \angle CBK + \frac{\widehat{AK}}{2} = 180^\circ - \angle DBA + \angle ADK = \\ &= \angle DBV + \angle BDV = 180^\circ - \angle AVD.\end{aligned}$$

Consequently $\angle CXA + \angle AVD = 180^\circ$. This equality means that the point X belongs to the circumcircle of $\triangle ACV$. For this reason $\angle XAB = \angle XCD$. The second equality could be obtained in the following way:

$$\angle XBC = \frac{\widehat{XC}}{2} = \angle XKC = 180^\circ - \angle AKX = \angle XDA.$$

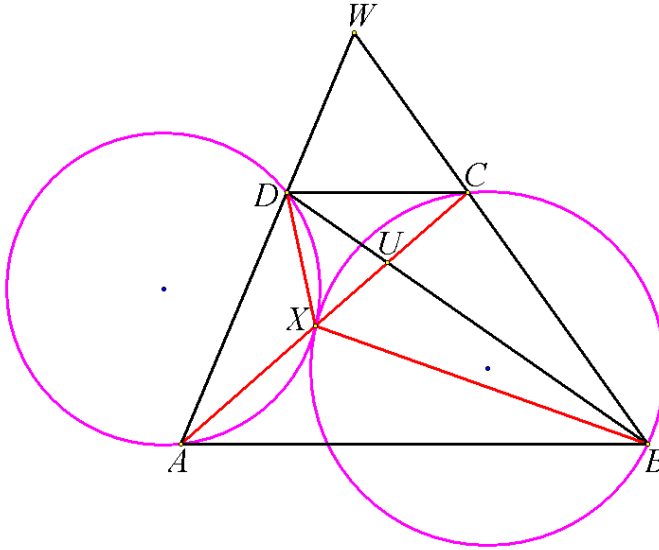
The assertion for the other couple of circles under the condition $AD \cap BC = W$ could be proven in the same way.





If $AB \parallel CD$ and the second common point of $k(BCK)$ and $k(DAK)$ is X , as in the case just considered, we obtain $\angle CXA = 180^\circ - \angle DBA + \angle BDC = 180^\circ$. This means that the points C , X and A are collinear, i.e. the point X is on the diagonal AC . Consequently, the common points X and K of the circles $k(BCK)$ and $k(DAK)$ are on the diagonal AC . But the circle $k(BCK)$ intersects the line AC in the point C . Since a circle and a line could have no more than two common points, then the point X coincides with K . Therefore, in this case the circles $k(BCK)$ and $k(DAK)$ are tangent at the point X .

Analogously, if $AD \parallel BC$, the circles $k(BCK')$ and $k(DAK')$ are tangent at the point $X \equiv K'$.



Solution of problem 6: If $ABCD$ is a parallelogram, then the equality $AB \cdot CD = BC \cdot DA$ is satisfied only in the case when $ABCD$ is a rhombus. Since the point X is the intersection point of its diagonals, then $\angle BXA + \angle DXC = 90^\circ + 90^\circ = 180^\circ$. Let now $ABCD$ is such, that two of its sides at least are not parallel to each other. For definiteness let $AB \cap CD = V$. Since (according to lemma 5) the point X belongs to the circles $k(BCK)$ and $k(DAK)$, then $\angle DXA + \angle BXC = \angle AKD + \angle BKC$. From the conclusion we obtain, that $\angle DXA + \angle BXC = 180^\circ$. But $(\angle BXA + \angle DXC) + (\angle DXA + \angle BXC) = 360^\circ$. Therefore $\angle BXA + \angle DXC = 180^\circ$. If we consider the point X as a common point of the circles $k(BCK')$ and $k(DAK')$, we deduce the desired equality as a consequence of lemma 5 and the conclusion in the following way:

$$\angle BXA + \angle DXC = \angle BK'A + \angle CK'D = 180^\circ.$$

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ЗАДАЧА 6. ОТ IMO'2018

Резюме. Международната олимпиада по математика е едно от уважаваните събития и едно от най-старите международни образователни и научни състезания. Тя е най-голямата, най-старата и най-престижната научна олимпиада за гимназиални ученици. 59-ото издание на събитието се проведе в Клуж-Напока, Румъния, в периода 3 – 14 юли 2018. Настоящата статия е посветена на шестата задача от темата. Предложен е подробен анализ на задачата с методологически характер, който ще бъде полезен за ученици и учители по време на подготовката за бъдещи участия в математически състезания.

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