

INTERESTING PROOFS OF SOME ALGEBRAIC INEQUALITIES

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Abstract. The paper considers interesting proofs of two algebraic inequalities.

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First, we will give the proof of the next algebraic inequality with radicals:

$$\sqrt{x^2+3} + \sqrt{y^2+3} + \sqrt{xy+3} \geq 6; \quad (x, y \geq 0, x+y=2). \quad (1)$$

Proof: We will prove the two inequalities:

$$\sqrt{x^2+3} + \sqrt{y^2+3} \geq \sqrt{19-3xy} \quad (2)$$

and

$$\sqrt{19-3xy} + \sqrt{xy+3} \geq 6, \quad (3)$$

where $x, y \geq 0$ and $x+y=2$.

1^0 . We have $0 \leq xy = t \leq 1$ because $\frac{x+y}{2} \geq \sqrt{xy}$, i.e. $1 \geq \sqrt{xy} \Rightarrow 0 \leq xy \leq 1$. On the other hand

$$x^2 + y^2 = (x+y)^2 - 2xy = 4 - 2t.$$

After squaring, we get from (2):

$$x^2 + 3 + y^2 + 3 + 2\sqrt{x^2+3} \cdot \sqrt{y^2+3} \geq 19 - 3xy$$

$$\Leftrightarrow (x+y)^2 - 2xy + 6 + 2\sqrt{x^2y^2 + 3(x^2+y^2) + 9} \geq 19 - 3xy$$

$$\Leftrightarrow 4 - 2t + 6 + 2\sqrt{t^2 + 3(4-2t) + 9} \geq 19 - 3t$$

$$\Leftrightarrow 2\sqrt{t^2 - 6t + 21} \geq 9 - t$$

$$\Leftrightarrow 4(t^2 - 6t + 21) \geq 81 - 18t + t^2$$

$$\Leftrightarrow 3t^2 - 6t + 3 \geq 0$$

$$\Leftrightarrow t^2 - 2t + 1 \geq 0$$

$$\Leftrightarrow (t-1)^2 \geq 0,$$

which is obvious. Equality holds true for $t = 1$, i.e. $xy = 1 \Rightarrow y = \frac{1}{x} \Rightarrow x + \frac{1}{x} = 2 \Rightarrow (x-1)^2 = 0 \Rightarrow x = 1$ and $y = 1$. Thus, the inequality (2) is proved.

2⁰. Now, we will prove the inequality (3), i.e.:

$$\sqrt{19-3xy} + \sqrt{xy+3} \geq 6$$

$$\Leftrightarrow 19-3xy + xy + 3 + 2\sqrt{19-3xy} \cdot \sqrt{xy+3} \geq 36$$

$$\Leftrightarrow 2\sqrt{10xy+57-3x^2y^2} \geq 14+2xy$$

$$\Leftrightarrow \sqrt{10t+57-3t^2} \geq 7+t$$

$$\Leftrightarrow 10t+57-3t^2 \geq 49+14t+t^2$$

$$\Leftrightarrow 4t^2+4t-8 \leq 0$$

$$\Leftrightarrow t^2+t-2 \leq 0$$

$$\Leftrightarrow (t-1)(t+2) \leq 0,$$

which is true because $0 \leq t \leq 1$, i.e. $t-1 \leq 0$. Equality holds, for $t = 1$, i.e. for $x = y = 1$. Thus, the inequality (3) is also proved.

The given inequality (1) follows from (2) and (3).

In the proof above a very important question arises: How to come to the inequality (2)? We will give an answer to this question in the proof of the next inequality:

$$\sqrt{x^2+8} + \sqrt{y^2+8} + \sqrt{xy+8} \geq 9; \quad (x, y \geq 0, x+y=2). \quad (4)$$

Proof: We will consider the inequality:

$$\sqrt{x^2+8} + \sqrt{y^2+8} \geq \sqrt{\alpha - \beta xy}; \quad (\alpha, \beta \in \mathbb{Q}) \quad (5)$$

$$\Leftrightarrow x^2 + 8 + y^2 + 8 + 2\sqrt{x^2 + 8} \cdot \sqrt{y^2 + 8} \geq \alpha - \beta xy$$

$$\Leftrightarrow x^2 + y^2 + 16 + 2\sqrt{x^2 y^2 + 8(x^2 + y^2) + 64} \geq \alpha - \beta xy$$

$$\Leftrightarrow 4 - 2t + 16 + 2\sqrt{t^2 + 8(4 - 2t) + 64} \geq \alpha - \beta t$$

$$\Leftrightarrow 2\sqrt{t^2 - 16t + 96} \geq \alpha - 20 + (2 - \beta)t. \quad (6)$$

Equality holds in (5) for $x = y = 1$, and we get from (5) that $\sqrt{\alpha - \beta} = 6$, i.e. $\alpha - \beta = 36$.

The inequality (6) is equivalent now with the inequality:

$$2\sqrt{t^2 - 16t + 96} \geq 16 + \beta + (2 - \beta)t$$

$$\Leftrightarrow 4(t^2 - 16t + 96) \geq (16 + \beta)^2 + 2(16 + \beta)(2 - \beta)t + (2 - \beta)^2 t^2.$$

Let now

$$P(t) = [4 - (2 - \beta)^2]t^2 + [-64 - 2(16 + \beta)(2 - \beta)]t + 384 - (16 + \beta)^2 \geq 0.$$

Since $P(t)$ is the minimal value of the polynomial in the interval under consideration, then it follows that $P'(t) = 0$, i.e.:

$$P'(t) = 2[4 - (2 - \beta)^2]t - 64 - 2(16 + \beta)(2 - \beta)$$

$$\Rightarrow P'(1) = 8 - 2(2 - \beta)^2 - 64 - 2(16 + \beta)(2 - \beta) = 0$$

$$\Rightarrow 4 - (4 - 4\beta + \beta^2) - 32 - (32 - 14\beta - \beta^2) = 0$$

$$\Rightarrow 4 - 4 + 4\beta - \beta^2 - 32 - 32 + 14\beta + \beta^2 = 0$$

$$\Rightarrow 18\beta - 64 = 0$$

$$\Rightarrow \beta = \frac{32}{9},$$

and from here (because $\alpha - \beta = 36$) we get $\alpha = \frac{356}{9}$.

We will now prove the inequality:

$$\sqrt{x^2 + 8} + \sqrt{y^2 + 8} \geq \sqrt{\frac{356}{9} - \frac{32}{9}xy} \quad (7)$$

$$\Leftrightarrow x^2 + 8 + y^2 + 8 + 2\sqrt{x^2 + 8} \cdot \sqrt{y^2 + 8} \geq \frac{356}{9} - \frac{32}{9}xy$$

$$\Leftrightarrow 4 - 2t + 16 + 2\sqrt{t^2 - 16t + 96} \geq \frac{356}{9} - \frac{32}{9}t$$

$$\Leftrightarrow 9\sqrt{t^2 - 16t + 96} \geq 88 - 7t$$

$$\Leftrightarrow 81(t^2 - 16t + 96) \geq 7744 - 1232t + 49t^2$$

$$\Leftrightarrow 32t^2 - 64t + 32 \geq 0$$

$$\Leftrightarrow t^2 - 2t + 1 \geq 0$$

$$\Leftrightarrow (t - 1)^2 \geq 0 ,$$

which is obvious. Equality holds for $t = 1$, i.e. $x = y = 1$. Thus, the inequality (7) is proved.

Finally, we will prove the inequality:

$$\sqrt{\frac{356}{9} - \frac{32}{9}xy} + \sqrt{xy + 8} \geq 9 \quad (8)$$

$$\Leftrightarrow \frac{356}{9} - \frac{32}{9}xy + xy + 8 + 2\sqrt{\frac{356}{9} - \frac{32}{9}xy} \cdot \sqrt{xy + 8} \geq 81$$

$$\Leftrightarrow 356 - 32xy + 9xy + 72 + 6\sqrt{(356 - 32xy)(xy + 8)} \geq 729$$

$$\Leftrightarrow 428 - 23t + 6\sqrt{100t - 32t^2 + 2848} \geq 729$$

$$\Leftrightarrow 6\sqrt{100t - 32t^2 + 2848} \geq 301 + 23t$$

$$\Leftrightarrow 36(100t - 32t^2 + 2848) \geq 90601 + 529t^2 + 13846t$$

$$\Leftrightarrow 1681t^2 + 10246t - 11927 \leq 0$$

$$\Leftrightarrow (t - 1)(1681t + 11927) \leq 0 ,$$

which is true because $0 \leq t \leq 1$, i.e. $t - 1 \leq 0$. Equality holds for $t = 1$, i.e. $x = y = 1$. Thus, the inequality (8) is proved.

Now, the given inequality (4) follows from (7) and (8).

We leave the following inequality to the reader:

$$\sqrt{x^2 + 15} + \sqrt{y^2 + 15} + \sqrt{xy + 15} \geq 12; \quad (x, y \geq 0, x + y = 2). \quad (9)$$

For the proof of this inequality one could consider two other inequalities:

$$\sqrt{x^2 + 15} + \sqrt{y^2 + 15} \geq \sqrt{\alpha - \beta xy}; \quad (\alpha, \beta \in \mathbb{Q}) \quad (10)$$

and

$$\sqrt{\alpha - \beta xy} + \sqrt{xy + 15} \geq 12. \quad (11)$$

Finally, we formulate a generalised version:

$$\sqrt{x^2 + n^2 - 1} + \sqrt{y^2 + n^2 - 1} + \sqrt{xy + n^2 - 1} \geq 3n \quad (12)$$

where $x, y \geq 0, x + y = 2$ and $n \in \mathbb{N}, (n \geq 2)$.

Much work but at the same much pleasure. This is Mathematics.

NOTES/БЕЛЕЖКИ

1. AOPS-Art of ProblemSolving (web page).

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