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GENERALIZED TRIGONOMETRIC IDENTITIES

Valentin Vankov Iliev
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences (Bulgaria)

Abstract. In this short note we present the use of a form of binary symmetry in trigonometry in order to present naturally grouped trigonometric identities in a compact way. This approach is more conceptual and, in particular, replaces a tedious consideration of similar cases.

Keywords: generalized trigonometric identity; binary symmetry; shortening the exposition

1. Notation, Introduction

1.1. Notation

The following notation will be frequently used in this paper:
$\mathbb{Z}_2 = \{0, 1\}$: the field of two elements. We also consider the external law of composition $0.x = 0$ and $1.x = x$, on the set of real numbers $x \in \mathbb{R}$ with the field $\mathbb{Z}_2$ as operating set. This law satisfies the properties $(ks).x = k.(s.x)$, $k.(x + y) = k.x + k.y$, and $k.(xy) = (k.x)y = x(k.y)$;
When $-1 \in \mathbb{R}$ and $k \in \mathbb{Z}_2$, the power $(-1)^k$ is well defined;
For $0, 1 \in \mathbb{Z}_2$ and any $\alpha \in \mathbb{R}$ we denote: $\operatorname{tr}_0 \alpha = \cos \alpha$, $\operatorname{tr}_1 \alpha = \sin \alpha$.

1.2. Introduction

An expression of the form $I(k_1, k_2, \ldots, k_n) = 0$, $k_i \in \mathbb{Z}_2$, where the left hand side is a polynomial with real coefficients and "unknowns" of the form $\operatorname{tr}_{f(k_1,k_2,\ldots,k_n)}(\alpha, \beta, \ldots)$, $(-1)^{g(k_1,k_2,\ldots,k_n)}$, where $\alpha, \beta, \ldots \in \mathbb{R}$, and with polynomials $f(k_1, k_2, \ldots, k_n)$, $g(k_1, k_2, \ldots, k_n)$ over the field $\mathbb{Z}_2$, is said to be a *generalized trigonometric identity* if for any vector $(k_1, k_2, \ldots, k_n)$ from $\mathbb{Z}_2$-vector space $(\mathbb{Z}_2)^n$ it is a trigonometric identity. The last statement can also be read as a hidden definition of a *trigonometric identity* as a specialization of a generalized one. It turns out that

$$(I(k_1, k_2, \ldots, k_n) = 0)_{(k_1,k_2,\ldots,k_n)\in(\mathbb{Z}_2)^n} \tag{1.2.1}$$

is a family of naturally grouped trigonometric identities. For example, the generalized trigonometric identity

$$\operatorname{tr}_k(\theta) + (-1)^t \operatorname{tr}_k(\varphi) = 2(-1)^{(k+1)t} \operatorname{tr}_{k+t}\left(\frac{\theta+\varphi}{2}\right) \operatorname{tr}_t\left(\frac{\theta-\varphi}{2}\right), k, t \in \mathbb{Z}_2, \tag{1.2.2}$$

produces the four well known trigonometric identities

$$\begin{aligned}
k=0, t=0 &: \cos\theta + \cos\varphi = 2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right),\\
k=0, t=1 &: \cos\theta - \cos\varphi = -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right),\\
k=1, t=0 &: \sin\theta + \sin\varphi = 2\sin\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right),\\
k=1, t=1 &: \sin\theta - \sin\varphi = 2\cos\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right).
\end{aligned} \tag{1.2.3}$$

Note that the involution $(k,t) \mapsto (k+1,t)$ of the $\mathbb{Z}_2$-vector space $(\mathbb{Z}_2)^2$ induces an involution on the above set of four trigonometric identities, which interchanges sine functions and cosine functions to the left-hand side as a mirror symmetry.

Below we present several generalized trigonometric identities and their application for solving some generalized trigonometric equations. The author uses this kind of $\mathbb{Z}_2$-symmetry in trigonometry in his paper (Iliev, 2025). This approach shortens the exposition significantly because it avoids considering of several similar cases. In Section 2 we present an example of how to construct the generalized trigonometric identity (1.2.2) which produces the family (1.2.3) of trigonometric identities.

All citations are taken from the famous book (Gelfand & Saul, 2001) and are included for the sake of completeness.

2. Construction of Generalized Trigonometric Identities

It turns out that given a family (1.2.1) of 2^n "naturally" grouped trigonometric identities we can find their "generalized source" $I(k_1, k_2, \ldots, k_n) = 0$ by solving a system of linear equations over the field $\mathbb{Z}_2$. As an example of this procedure we construct the generalized trigonometric identity (1.2.2), starting from the group (1.2.3) of trigonometric identities. The obvious general draft of (1.2.2) has the form

$$\operatorname{tr}_k(\theta)+(-1)^t \operatorname{tr}_k(\varphi) = 2(-1)^{a(k,t)}\, tr_{b(k,t)}\left(\frac{\theta+\varphi}{2}\right) \operatorname{tr}_{c(k,t)}\left(\frac{\theta-\varphi}{2}\right), k,t \in \mathbb{Z}_2,$$

with unknown polynomials $a\,(k,t)$, $b\,(k,t)$, and $c\,(k,t)$ from the ring $\mathbb{Z}_2[k,t]$. Using the trial and error method, we begin with linear polynomials $a = a_1k + a_2t$, $b = b_1k + b_2t$,and $c = c_1k + c_2t$, whose coefficients $a_i, b_i, c_i \in \mathbb{Z}_2$ have to be determined. Taking into account (1.2.3), we obtain first that $0 = a\,(0,0) = 0a_1 + 0a_2$, $1 = a\,(0,1) = 0a_1 + a_2$, $0 = a\,(1,0) = a_1 + 0a_2$, and $0 = a\,(1,1) = a_1 + a_2$, which is a contradiction. Therefore $a\,(k,t)$ cannot be linear and we try with a quadratic polynomial: $a\,(k,t) = a_1k + a_2t + a_3kt$. We have $0 = a\,(0,0) = 0a_1 + 0a_2 + 0a_3$, $1 = a\,(0,1) = 0a_1 + a_2 + 0a_3$, $0 = a\,(1,0) = a_1+0a_2+0a_3$, and $0 = a\,(1,1) = a_1+a_2+a_3$. In other words, $a_1 = 0$, $a_2 = 1$, $a_3 = 1$, that is, $a = (k+1)t$. Analogously, $0 = b\,(0,0) = 0b_1 + 0b_2$, $1 = b\,(0,1) = 0b_1 + b_2$, $1 = b\,(1,0) = b_1 + 0b_2$, and $0 = b\,(1,1) = b_1 + b_2$, hence $b = k + t$. Finally, in the same way we obtain that $c = t$.

3. Some Main Identities

3.1. Odd/Even Identities

For any $s, t \in \mathbb{Z}_2$ and $\theta \in \mathbb{R}$ one has

$$\operatorname{tr}_s((-1)^t\theta) = (-1)^{st} \operatorname{tr}_s(\theta),$$

see (Gelfand & Saul, 2001, Ch. 4, 5). Note that for $s = 0, t = 1$ we obtain $\cos(-\alpha) = \cos\alpha$ and for $s = 1, t = 1$ we have $\sin(-\alpha) = -\sin\alpha$.

3.2. Angle Sum and Difference Identities

For any $s, t \in \mathbb{Z}_2$ and $\alpha, \beta \in \mathbb{R}$ one has

$$\operatorname{tr}_s(\alpha + (-1)^t\beta) = \operatorname{tr}_s(\alpha)\operatorname{tr}_0(\beta) + (-1)^{s+t+1}\operatorname{tr}_{s+1}(\alpha)\operatorname{tr}_1(\beta),$$

see (Gelfand & Saul, 2001, Ch. 6, 2; Ch. 7, 2). Note that the specialization $s = 0, t = 0$ yields the trigonometric identity $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, the specialization $s = 0, t = 1$ yields $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, the specialization $s = 1, t = 0$ yields $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, and the specialization $s = 1, t = 1$ yields $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$. Below we omit similar remarks.

In particular, for any $s \in \mathbb{Z}_2$ and $\beta \in \mathbb{R}$ one has

$$\operatorname{tr}_{s+1}(\frac{\pi}{2} - \beta) = \operatorname{tr}_s(\beta), \operatorname{tr}_{s+1}(\frac{3\pi}{2} - \beta) = -\operatorname{tr}_s(\beta).$$

3.3. Product-to-Sum Identities

For any $s, t \in \mathbb{Z}_2$ and $\alpha, \beta \in \mathbb{R}$ one has

$$\operatorname{tr}_s \alpha \operatorname{tr}_t \beta = \frac{1}{2}(\operatorname{tr}_{s+t}(\alpha + (-1)^{s+t+1}\beta) + (-1)^t \operatorname{tr}_{s+t}(\alpha + (-1)^{s+t}\beta)),$$

see (Gelfand & Saul, 2001, Ch. 7, 9).

3.4. Sum-to-Product Identities

We change the variables in the identity from 3.3 by the rules $\theta = \alpha + (-1)^{s+t+1}\beta$, $\varphi = \alpha + (-1)^{s+t}\beta$. Substituting $k = s + t$ and using 3.1, we obtain that for any $k, t \in \mathbb{Z}_2$ and $\theta, \varphi \in \mathbb{R}$ one has

$$\operatorname{tr}_k(\theta) + (-1)^t \operatorname{tr}_k(\varphi) = 2(-1)^{(k+1)t} \operatorname{tr}_{k+t}\left(\frac{\theta+\varphi}{2}\right) \operatorname{tr}_t\left(\frac{\theta-\varphi}{2}\right),$$

see (Gelfand & Saul, 2001, Ch. 7, 10).

4. Solutions of Some Generalized Trigonometric Equations

4.1. The Equation $\operatorname{tr}_k(\theta) = 0$*,* $\theta \in I$*, where* $I \subset \mathbb{R}$ *is a half-open interval of length* 2π

Lemma 4.1.1 *Let $c \in \mathbb{R}$. If m_c is the minimal integer such that $c + m_c\pi \in I$, then $c + (m_c + 1)\pi \in I$.*

Proof: Let a and b, with $a < b$, be the endpoints of the interval I. If $a \in I$, then $c + (m_c - 1)\pi < a$ and $c + (m_c + 1)\pi < a + 2\pi = b$. If $a \notin I$, then $c + (m_c - 1)\pi \leq a$ and $c + (m_c + 1)\pi \leq a + 2\pi = b$.

The solution of the equation stated in the title is

$$\theta = (k + 1).(\frac{\pi}{2} + m_{\frac{\pi}{2}}\pi) + k.(m_0\pi) + n.\pi, n \in \mathbb{Z}_2.$$

Remark 4.1.2 If $I = [0, 2\pi)$ or $I = [-\frac{\pi}{4}, \frac{7\pi}{4})$, then $m_{\frac{\pi}{2}} = 0$ and $m_0 = 0$. If $I = (-\pi, \pi]$ or $I = [-\frac{3\pi}{4}, \frac{5\pi}{4})$,, then $m_{\frac{\pi}{2}} = -1$ and $m_0 = 0$.

4.2. The Equation $\operatorname{tr}_k(\theta) + (-1)^t \operatorname{tr}_s(\varphi) = 0$*,* $\theta, \varphi \in [0, 2\pi)$

Below we apply the identities from 3.2, 3.4, and Remark 4.1.2. Moreover, we note that $\frac{\theta+\varphi}{2} \in [0, 2\pi)$, $\frac{\theta-\varphi}{2} \in (-\pi, \pi]$.

Case 1. $s = k$

The equation $\operatorname{tr}_{k+t}\left(\frac{\theta+\varphi}{2}\right) = 0$ has solutions $\frac{\theta+\varphi}{2} = (k + t + 1).\frac{\pi}{2} + n.\pi$, $n \in \mathbb{Z}_2$, or, equivalently, $\theta + \varphi = (k + t + 1).\pi + n.(2\pi)$, $n \in \mathbb{Z}_2$. Moreover, the equation $\mathrm{tr}_t\left(\frac{\theta-\varphi}{2}\right)=0$ has solutions $\frac{\theta-\varphi}{2}=(t+1).(-\frac{\pi}{2})+n.\pi$, $n\in\mathbb{Z}_2$, or, equivalently, $\theta-\varphi=-(t+1).\pi+n.(2\pi)$, $n\in\mathbb{Z}_2$.

Case 2. $s=k+1$

We obtain

$$\mathrm{tr}_k(\theta)+(-1)^t\,\mathrm{tr}_{k+1}(\varphi)=\mathrm{tr}_k(\theta)+(-1)^t\,\mathrm{tr}_k(\frac{\pi}{2}-\varphi)=$$
$$2(-1)^{(k+1)t}\,\mathrm{tr}_{k+t}\left(\frac{\theta-\varphi}{2}+\frac{\pi}{4}\right)\mathrm{tr}_t\left(\frac{\theta+\varphi}{2}-\frac{\pi}{4}\right).$$

Moreover, $\frac{\theta+\varphi}{2}-\frac{\pi}{4}\in[-\frac{\pi}{4},\frac{7\pi}{4})$, $\frac{\theta-\varphi}{2}+\frac{\pi}{4}\in(-\frac{3\pi}{4},\frac{5\pi}{4}]$.

The equation $\mathrm{tr}_{k+t}\left(\frac{\theta-\varphi}{2}+\frac{\pi}{4}\right)=0$ has solutions $\frac{\theta-\varphi}{2}+\frac{\pi}{4}=(k+t+1).(-\frac{\pi}{2})+n.\pi$, $n\in\mathbb{Z}_2$, or, equivalently, $\theta-\varphi=-(k+t+1).\pi+n.(2\pi)-\frac{\pi}{2}$, $n\in\mathbb{Z}_2$.

The equation $\mathrm{tr}_t\left(\frac{\theta+\varphi}{2}-\frac{\pi}{4}\right)=0$ has solutions $\frac{\theta+\varphi}{2}-\frac{\pi}{4}=(t+1).\frac{\pi}{2}+n.\pi$, $n\in\mathbb{Z}_2$, or, equivalently, $\theta+\varphi=(t+1).\pi+n.(2\pi)+\frac{\pi}{2}$, $n\in\mathbb{Z}_2$.

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✉ **Valentin Vankov Iliev**
ORCID iD: 0000-0002-5271-8810
Section of Algebra and Logic
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
8, G. Bonchev Str., 1113 Sofia, Bulgaria
E-mail: viliev@math.bas.bg