

FIVE WAYS TO SOLVE A PROBLEM FOR A TRIANGLE

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Abstract. A triangle with the property $3\alpha + 2\beta = 180^\circ$ is considered in the paper. Five different ways are proposed to prove that $a^2 + bc = c^2$. The paper is of methodological character.

Keywords: triangle, problem solving, trigonometry, similarity.

To solve a mathematical problem in several ways is challenging and creative but also instructive and useful for a deep examination of the problem history and content to discover the initial idea of its author and to find the potential of the possible applications. This is exceptionally important for talented students who touch various mathematical domains by the different solutions, thus increasing their knowledge and capacity. Consider the following problem:

Using the standard notations for $\triangle ABC$, prove that $a^2 + bc = c^2$ if $3\alpha + 2\beta = 180^\circ$. We will give five solutions of this problem.

Solution 1. Accounting for the relations $3\alpha + 2\beta = 180^\circ$ and $\alpha + \beta + \gamma = 180^\circ$, we have $\gamma = 180^\circ - (\alpha + \beta) = 3\alpha + 2\beta - (\alpha + \beta) = 2\alpha + \beta$. It follows that $\gamma > \beta$ and hence $c > b$. Take now a point D on the side AB such that $\angle BCD = \angle ABC = \beta$ (Fig. 1). This implies that $BD = AD$, i.e.

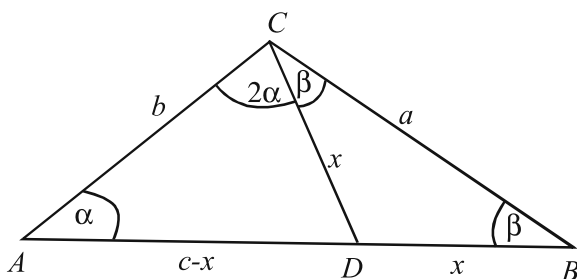


Fig. 1

$\triangle BCD$ is isosceles. If the lengths of the legs are denoted by x , then $AD = c - x$. By the law of sines and the law of cosines for $\triangle ADC$ we get:

$$\frac{x}{\sin \alpha} = \frac{c-x}{\sin 2\alpha} \text{ and } x^2 = b^2 + (c-x)^2 - 2b(c-x)\cos \alpha.$$

Since $\sin 2\alpha = 2 \sin \alpha \cos \alpha$, the first equality implies that $\cos \alpha = \frac{c-x}{2x}$. Now, the second equality becomes $x^2 = b^2 + (c-x)^2 - 2b(c-x)\frac{c-x}{2x}$ and consequently $x = \frac{c^2}{b+2c}$. The law of sines for $\triangle BCD$ and the law of cosines for $\triangle ABC$ give

$$\frac{x}{\sin \beta} = \frac{a}{\sin(180^\circ - 2\beta)} \text{ and } b^2 = a^2 + c^2 - 2ac \cos \beta, \text{ respectively.}$$

Since $\sin(180^\circ - 2\beta) = \sin 2\beta = 2 \sin \beta \cos \beta$, it follows from the first equality that $\cos \beta = \frac{a}{2x}$. Now, the second equality becomes $b^2 = a^2 + c^2 - 2ac \frac{a}{2x}$ and consequently $x = \frac{a^2 c}{a^2 + c^2 - b^2}$. We have obtained 2 expressions for x , which give:

$$\frac{c^2}{b+2c} = \frac{a^2 c}{a^2 + c^2 - b^2}.$$

It is easy to check that the last is equivalent to $a^2 + bc = c^2$.

Solution 2. Take a point E on the side AB such that $\angle ACE = \angle CAE = \alpha$ (Fig.2).

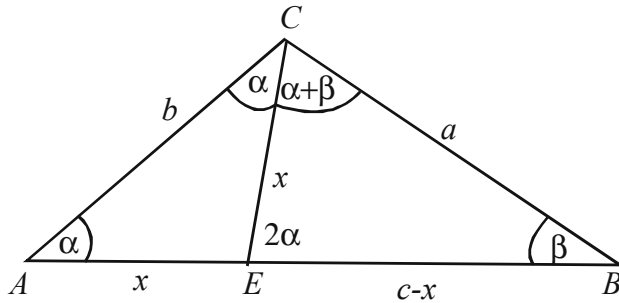


Fig. 2

Now $\triangle AEC$ is isosceles and if $AE = CE = x$, then $BE = c - x$. Applying the Mollweid's formula for $\triangle BCE$, we get:

$$\frac{(c-x)+x}{a} = \frac{\cos \frac{(\alpha+\beta)-\beta}{2}}{\sin \frac{2\alpha}{2}} = \frac{\cos \frac{\alpha}{2}}{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} = \frac{1}{2 \sin \frac{\alpha}{2}}.$$

Thus $\frac{c}{a} = \frac{1}{2 \sin \frac{\alpha}{2}}$ and consequently $\sin \frac{\alpha}{2} = \frac{a}{2c}$. Further, apply the law of

cosines for $\triangle ABC$:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos \alpha = b^2 + c^2 - 2bc \left(1 - 2 \sin^2 \frac{\alpha}{2}\right) = \\ &= b^2 + c^2 - 2bc \left(1 - \frac{a^2}{2c^2}\right) = b^2 + c^2 - \frac{b}{c}(2c^2 - a^2). \end{aligned}$$

Now it is easy to check that the equality $a^2 = b^2 + c^2 - \frac{b}{c}(2c^2 - a^2)$ is equivalent to $a^2 + bc = c^2$, using that $c > b$.

Solution 3. Take a point F on the line AC (C is between A and F) such that $\angle CBF = \alpha + \beta$. (Fig. 3)

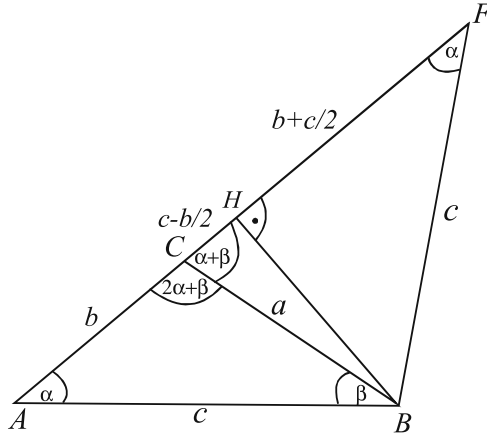


Fig. 3

Since $\angle BCF = \alpha + \beta$, then $\triangle BCF$ is isosceles. Also, we have that $\angle BFC = \alpha$ and it follows that $\triangle ABF$ is isosceles too, i.e. $AB = BF = CF = c$. If BH is the height of $\triangle ABF$ ($H \in AF$), then $AH = \frac{1}{2}AF = \frac{b+c}{2}$ and consequently

$CH = \frac{b+c}{2} - b = \frac{c-b}{2}$, using again that $c > b$. Finally, apply the Pithagoras theorem to the right triangles $\triangle BHF$ and $\triangle BCH$. We have:

$$BH^2 = c^2 - \left(\frac{b+c}{2}\right)^2 \text{ and } BH^2 = a^2 - \left(\frac{c-b}{2}\right)^2.$$

Now, check that the equality $c^2 - \left(\frac{b+c}{2}\right)^2 = a^2 - \left(\frac{c-b}{2}\right)^2$ is equivalent to $a^2 + bc = c^2$.

Solution 4. Applying the Stewart's theorem to $\triangle ABF$ (Fig. 3), we have:

$AF \cdot (AC \cdot CF + BC^2) = AB^2 \cdot CF + BF^2 \cdot AC$, which gives, that

$$(b+c)(bc+a^2) = c^2c + c^2b, \text{ i.e. } (b+c)(bc+a^2) = (c+b)c^2.$$

The last is equivalent to $a^2 + bc = c^2$.

Solution 5. Let $G \in AB$ be such point that $\angle BCG = \angle BAC = \alpha$. (Fig. 4)

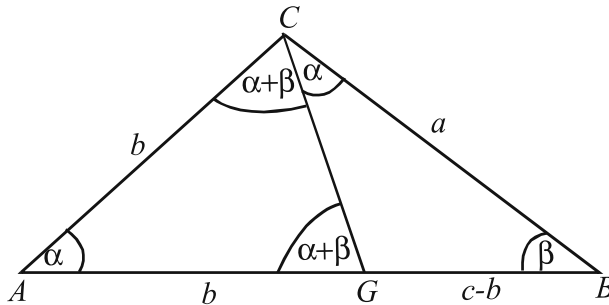


Fig. 4

It follows that $\angle AGC = \angle ACG = \alpha + \beta$ and $\triangle AGC$ is isosceles. Hence, $GB = c - b$. Since $\angle GCB = \alpha$, the triangles ABC and CBG are similar. Thus, $\frac{a}{c-b} = \frac{c}{a}$, which is equivalent to $a^2 + bc = c^2$.

Using trigonometry (the law of sines, the law of cosines and the Mollweid's formula), solution 1 and solution 2 are rather complicated. On the contrary, solutions 3–5 seem to be shorter, avoiding trigonometry. They apply the Pythagoras theorem, the Stewart's theorem and similarity of triangles, which are deep mathematical facts. Maybe, this makes them more interesting and creative.

For exercise we recommend the following problems:

1. Given is $\triangle ABC$. Prove that the relations $\alpha = 2\beta$ and $a^2 = b(b+c)$ are equivalent.

2. If the relation $\alpha - \beta = 90^\circ$ is valid for $\triangle ABC$, prove that $(a^2 - b^2)^2 = c^2(a^2 + b^2)$. (Such a triangle is called *pseudoright triangle*).
3. The angles α , β and γ of $\triangle ABC$ satisfy the equality $2\gamma = \alpha - \beta$. Prove that $c^2 = a(a - b)$.
4. The proportion $\alpha : \beta : \gamma = 4 : 2 : 1$ is valid for $\triangle ABC$. Prove that $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$.
5. It is given for an acute $\triangle ABC$ that $\angle ACB = 2\angle ABC$. If $D \in BC$ is such a point that $2\angle BAD = \angle ABC$, prove that $\frac{1}{BD} = \frac{1}{AB} + \frac{1}{AC}$.
6. If the equality $\alpha = 3\beta$ is valid for $\triangle ABC$, prove that $bc^2 = (a - b)^2(a + b)$.

REFERENCES

1. Arslanagić, Š. (2004). *Matematika za nadarene*. Sarajevo: Bosanska riječ.
2. Blagojević, V. (2002). *Teoreme i zadaci iz planimetrije*. Sarajevo: Zavod za udžbenike i nastavna sredstva.
3. Marić, A. (1996). *Planimetrija – Zbirka riješenih zadataka*. Zagreb: Element.

ПЕТ НАЧИНА ЗА РЕШАВАНЕ НА ЕДНА ЗАДАЧА ЗА ТРИЪГЪЛНИК

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Резюме. В статията се разглежда триъгълник със свойството $3\alpha + 2\beta = 180^\circ$. Предложени са пет начина за доказване, че $a^2 + bc = c^2$. Статията е с методически характер.

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