# FIVE NEW PROOFS OF ONE TRIGONOMETRIC INEQUALITY IN THE TRIANGLE

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**Abstract.** The paper considers five proofs of the ineqaulity $\cos\alpha+\cos\beta+\cos\gamma\le\frac{3}{2}$ .
*Keywords*: trigonometric inequality, triangle, proof

Consider the following inequality:

$$\cos\alpha+\cos\beta+\cos\gamma\le\frac{3}{2}, \tag{1}$$

where α, β and γ are the interior angles of the triangle Δ*ABC*.

Twelve different proofs of this inequality are given in Arslanagić from 2008. We propose five new.

**Proof 1**. It follows by the cosine theorem, that:

$$\frac{b^2+c^2-a^2}{2bc}+\frac{c^2+a^2-b^2}{2ac}+\frac{a^2+b^2-c^2}{2ab}\le\frac{3}{2}$$
$$\Leftrightarrow a\left(b^2+c^2-a^2\right)+b\left(c^2+a^2-b^2\right)+c\left(a^2+b^2-c^2\right)\le 3abc$$
$$\Leftrightarrow a^2b+ab^2+b^2c+bc^2+a^2c+ac^2\le a^3+b^3+c^3+3abc\ ,$$

Thus, we come to the Schur's inequality (for $\lambda=1$). The equality holds in (1) iff $a=b=c$ and $\cos\alpha=\cos\beta=\cos\gamma=\frac{1}{2}$, respectively, i.e. $\alpha=\beta=\gamma=\frac{\pi}{3}$ (equilateral triangle).

**Proof 2**. It is shown in Proof 1, that

$$\cos\alpha+\cos\beta+\cos\gamma\le\frac{3}{2}\Leftrightarrow a^2b+ab^2+b^2c+bc^2+a^2c+ac^2\le a^3+b^3+c^3+3abc\ .$$

On the other hand $a^2b+ab^2+b^2c+bc^2+a^2c+ac^2=2s\left(s^2+r^2-2Rr\right)$,

$a^3+b^3+c^3=2s\left(s^2-3r^2-6Rr\right)$ and $abc=4Rrs$ , where $R$ and $r$ are the circum radius and the in-radius of $\Delta ABC$ , respectively, while $s$ is the semi-perimeter. We have:

$$cos\,\alpha+cos\,\beta+cos\,\gamma\le\frac{3}{2}\quad\Leftrightarrow 2s\left(s^2+r^2-2Rr\right)\le 2s\left(s^2-3r^2-6Rr\right)+12Rrs\;/:2s$$

$$\Leftrightarrow s^2+r^2-2Rr\le s^2-3r^2-6Rr+6Rr\quad\Leftrightarrow 2Rr\ge 4r^2\Leftrightarrow R\ge 2r\;.$$

The last is the well-known Euler's inequality and this ends the proof.

**Proof 3**. We have: $cos\,\alpha+cos\,\beta+cos\,\gamma=\frac{b^2+c^2-a^2}{2bc}+\frac{c^2+a^2-b^2}{2ac}+\frac{a^2+b^2-c^2}{2ab}=$

$$=\frac{a\left(b^2+c^2\right)-a^3+b\left(c^2+a^2\right)-b^3+c\left(a^2+b^2\right)-c^3}{2abc}$$

$$=\frac{(a+b+c)\left(a^2+b^2+c^2\right)-2\left(a^3+b^3+c^3\right)}{2abc}.$$

On the other hand:

$$a+b+c=2s,\;abc=4Rrs;\;a^2+b^2+c^2=2\left(s^2-r^2-4Rr\right);\;a^3+b^3+c^3=2s\left(s^2-3r^2-6Rr\right).$$

Thus,

$$cos\,\alpha+cos\,\beta+cos\,\gamma=\frac{2s\cdot 2\left(s^2-r^2-4Rr\right)-2\cdot 2s\left(s^2-3r^2-6Rr\right)}{2\cdot 4Rrs}$$

$$=\frac{s^2-r^2-4Rr-s^2+3r^2+6Rr}{2Rr}=1+\frac{r}{R}\cdot$$

It follows from the Euler's inequality $R\ge 2r$, that $\frac{r}{R}\le\frac{1}{2}$, hence $cos\,\alpha+cos\,\beta+cos\,\gamma\le\frac{3}{2}$.

**Proof 4**. For arbitrary real numbers $\alpha,\beta\in(0,\pi)$ we have:

$$\left(cos\,\alpha+cos\,\beta-1\right)^2+\left(sin\,\alpha-sin\,\beta\right)^2\ge 0$$

$$\Leftrightarrow cos^2\,\alpha+cos^2\,\beta+1+2\,cos\,\alpha\,cos\,\beta-2\,cos\,\alpha-2\,cos\,\beta+sin^2\,\alpha-2\,sin\,\alpha\,sin\,\beta+sin^2\,\beta\ge 0$$

$$\Leftrightarrow\left(sin^2\,\alpha+cos^2\,\alpha\right)+\left(sin^2\,\beta+cos^2\,\beta\right)+1+2\left(cos\,\alpha\,cos\,\beta-sin\,\alpha\,sin\,\beta-cos\,\alpha-cos\,\beta\right)\ge 0$$

$$\Leftrightarrow 1+1+1+2\,cos\left(\alpha+\beta\right)-2\,cos\,\alpha-2\,cos\,\beta\ge 0$$

$$\Leftrightarrow cos\,\alpha+cos\,\beta-cos\left(\alpha+\beta\right)\le\frac{3}{2},$$

But $\alpha+\beta+\gamma=\pi$, i.e. $\alpha+\beta=\pi-\gamma$, $cos(\alpha+\beta)=cos(\pi-\gamma)=-cos\gamma$, hence $cos\,\alpha+cos\,\beta+cos\,\gamma\le\frac{3}{2}$.

**Proof 5**. We use the following well-known expressions:

$$\cos 2x = \frac{1 - tg^2 x}{1 + tg^2 x} \quad \text{and} \quad \cos x = \frac{1 - tg^2 \frac{x}{2}}{1 + tg^2 \frac{x}{2}}.$$

It follows that: $\cos\alpha + \cos\beta + \cos\gamma \le \frac{3}{2} \Leftrightarrow \frac{1 - tg^2 \frac{\alpha}{2}}{1 + tg^2 \frac{\alpha}{2}} + \frac{1 - tg^2 \frac{\beta}{2}}{1 + tg^2 \frac{\beta}{2}} + \frac{1 - tg^2 \frac{\gamma}{2}}{1 + tg^2 \frac{\gamma}{2}} \le \frac{3}{2}$.

Use the substitutions: $u = tg\frac{\alpha}{2},\ v = tg\frac{\beta}{2},\ w = tg\frac{\gamma}{2};\ (u, v, w > 0)$ and the equality

$$tg\frac{\alpha}{2}tg\frac{\beta}{2} + tg\frac{\beta}{2}tg\frac{\gamma}{2} + tg\frac{\gamma}{2}tg\frac{\alpha}{2} = 1$$

We obtain $uv + vw + uw = 1$ and the inequality (1) is transformed to:

$$\frac{1 - u^2}{1 + u^2} + \frac{1 - v^2}{1 + v^2} + \frac{1 - w^2}{1 + w^2} \le \frac{3}{2}, \tag{2}$$

because $1 + u^2 = uv + vw + uw + u^2 = (u + v)(u + w)$ , $1 + v^2 = (v + w)(v + u)$ and $1 + w^2 = (w + u)(w + v)$. Thus

$$\frac{1 - u^2}{(u + v)(u + w)} + \frac{1 - v^2}{(v + w)(v + u)} + \frac{1 - w^2}{(w + u)(w + v)} \le \frac{3}{2}$$

$$\Leftrightarrow \frac{(1 - u^2)(v + w) + (1 - v^2)(u + w) + (1 - w^2)(u + v)}{(u + v)(v + w)(u + w)} \le \frac{3}{2}$$

$$\frac{v + w - u^2 v - u^2 w + u + w - uv^2 - v^2 w + u + v - uw^2 - vw^2}{(u + v)(v + w)(u + w)} \le \frac{3}{2}$$

$$\Leftrightarrow \frac{2(u + v + w) - \left[u^2(v + w) + v^2(u + w) + w^2(u + v)\right]}{(u + v)(v + w)(u + w)} \le \frac{3}{2}. \tag{3}$$

Using the arithmetic-geometric mean inequality, we have:

$$1 = uv + vw + uw \ge 3\sqrt[3]{u^2 v^2 w^2} \Leftrightarrow \sqrt{3} \ge 9uvw. \tag{4}$$

From the inequality

$$(x + y + z)^2 \ge 3(xy + yz + zx)\left(\Leftrightarrow \frac{1}{2}\left[(x - y)^2 + (y - z)^2 + (z - x)^2\right] \ge 0\right)$$

we get

$$u+v+w \geq \sqrt{3(uv+vw+uw)} = \sqrt{3\cdot 1} = \sqrt{3}\ . \qquad (5)$$

Further, $0<(u+v)(v+w)(u+w)=(u+v+w)(uv+vw+uw)-uvw=u+v+w-uvw$

and $0<u^2(v+w)+v^2(u+w)+w^2(u+v)=(u+v+w)(uv+vw+uw)-3uvw=u+v+w-3uvw$, hence:

$\dfrac{2(u+v+w)-(u+v+w-3uvw)}{u+v+w-uvw} \leq \dfrac{3}{2} \quad \Leftrightarrow \dfrac{u+v+w+3uvw}{u+v+w-uvw} \leq \dfrac{3}{2} \Leftrightarrow u+v+w \geq 9uvw$, which ends the proof.


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## ПЕТ НОВИ ДОКАЗАТЕЛСТВА НА ЕДНО ТРИГОНОМЕТРИЧНО НЕРАВЕНСТВО В ТРИЪГЪЛНИКА


**Резюме.** В статията се разглеждат пет доказателства на неравенството $\cos\alpha+\cos\beta+\cos\gamma \leq \dfrac{3}{2}$.



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