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Educational Issues
Въпроси на преподаването

ENHANCING VISUAL REASONING IN CALCULUS: INTRODUCING A NEW TYPOLOGY OF GRAPHICAL TASKS

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Abstract. This paper emphasizes the introduction of the innovative typology of tasks in calculus education to enhance students' visual reasoning skills. The proposed typology includes graphical tasks that require students to engage with images to derive meanings, justify solutions, and foster a deeper conceptual understanding of function derivatives. By integrating these tasks, we aim to address the imbalances in mathematics education regarding traditional approach, which often favors algebraic over graphical representation. This new typology will help students develop a well-rounded understanding, bridging the gap between symbolic manipulation and graphical interpretation.

Keywords: visualization; graphical tasks; calculus; derivative

1. Introduction

The role of visualization in the learning process and its impact on educational outcomes have occupied the attention of scientists from many fields, such as psychology, mathematics, mathematical education, etc., for many years, (Arcavi 2003; Clements 2014). Visualization is becoming increasingly significant due to the rapid development of digital technologies that provide the ability to present mathematical concepts through visual representations in a way that stimulates mathematical reasoning (Presmeg 2006).

One possible way to incorporate visual reasoning into mathematical education and curriculums in general, and particularly in calculus, is to introduce tasks where students will be compelled to use images to engage meanings, justify, and produce solutions. In the literature, for these kind of tasks (tasks with graphical content and/or requirements) different terms are used: graphical tasks, visualized tasks, visual reasoning tasks, etc. In this paper, we will use the term graphical tasks.

This paper will present an approach to task assignment based on the principles of visualization. The new typology of tasks that will be presented can illustrate the process and manner of implementing specific tasks designed for the purposes of teaching and learning in the field of calculus, and more precisely of function's derivatives.

2. Theoretical background

2.1. Visualization in Calculus

The Visual representations (pictures, diagrams, graphs) are essential for understanding mathematical concepts and communication. Mathematical visualization involves creating and manipulating representations (Hitt 1997). The connection between visual and symbolic representations can be challenging, especially for students focused on algebraic methods.

Calculus requires understanding both algebraic and graphical representations (Dunham & Osborne 1991). Conceptual understanding includes interpreting graphs (Vinner 1989; Zimmerman 1991), transitioning between representations (Habre & Abboud 2006), solving problems (Selden & Mason 1994), and understanding kinematic interpretations (Botzer & Yerushalmy 2008).

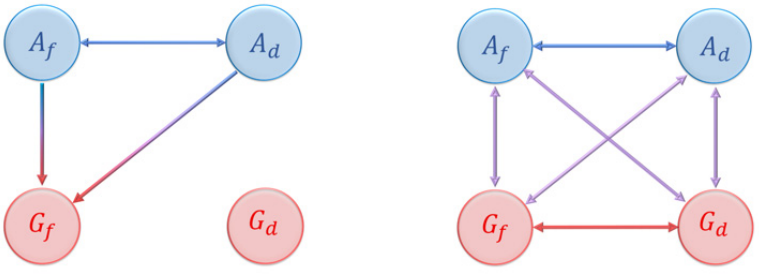
Teaching calculus is complex. Graphical representations pose difficulties for students (Gagatsis & Kyriakides 2003), while traditional methods prioritize algebraic approaches (Hitt 2002). Students often struggle with visual representations due to limited experience.

Understanding derivatives is crucial in calculus, involving functions, quotients, limits, and tangents. Students often face misconceptions (Asiala et al. 1997; Aspinwall et al. 1997; Berry & Nyman 2003; Orton 1983; Ubuz 2007) and difficulties with the limiting process (Biza et al. 2006; Vincent et al. 2015; Tall 2010).

2.2. Visual approach vs. the traditional one in Calculus

Traditional and visual approaches differ in methodology. To simplify representation-based comparisons, we introduce notations: Af (algebraic function), Ad (algebraic derivative), Gf (graphical function), and Gd (graphical derivative).

The notation Af refers to algebraic representations related to the concept of a function. The notation Gf refers to graphical representations related to the concept of a function. Similarly, Ad refers to algebraic, and Gd to graphical representations related to the concept of a function derivative. The traditional teaching concept and the visual approach concept, after these notations are introduced, can be presented schematically (Scheme 1).



Scheme 1. Transitions between Concepts and/or Representations – Comparative Overview of Traditional and Visual approach

Based on Scheme 1 a), algebraic representations dominate traditional teaching, while the graphical representations are under recognized. The traditional approach connects function and derivative concepts through only four transitions: $Af \rightarrow Ad$, $Ad \rightarrow Af$, $Af \rightarrow Gf$, and $Ad \rightarrow Gf$. Comparing both approaches, the visual method provides a richer learning environment by incorporating diverse representations and their connections.

3. Graphical tasks and the derivative of a function

To develop students' visual thinking and graphical understanding of functions and derivatives, the methodologically transformed content is designed, which connects the function's graph with its tangent (fig. 1 – 1st and 2nd graphical representation) and the tangent's slope with the derivative function's graph (fig. 1 – 3rd and 4th graphical representation).

A graphical understanding of derivatives requires knowledge of function graphs, tangents, slopes, and derivative function's graphs. Transitioning

between these representations involves referential connections. The first two transitions (1st→ 2nd, 2nd → 3rd) are visual, while linking the tangent's slope to the derivative graph (3rd → 4th) relies on algebraic knowledge: the tangent's slope coefficient ($k = tg\alpha = \frac{MP}{NP}$) and the derivative at a point ($f'(x_0) = k$). Understanding the tangent's slope as the derivative's value is cognitively demanding, requiring coordination between visual and symbolic abstraction.

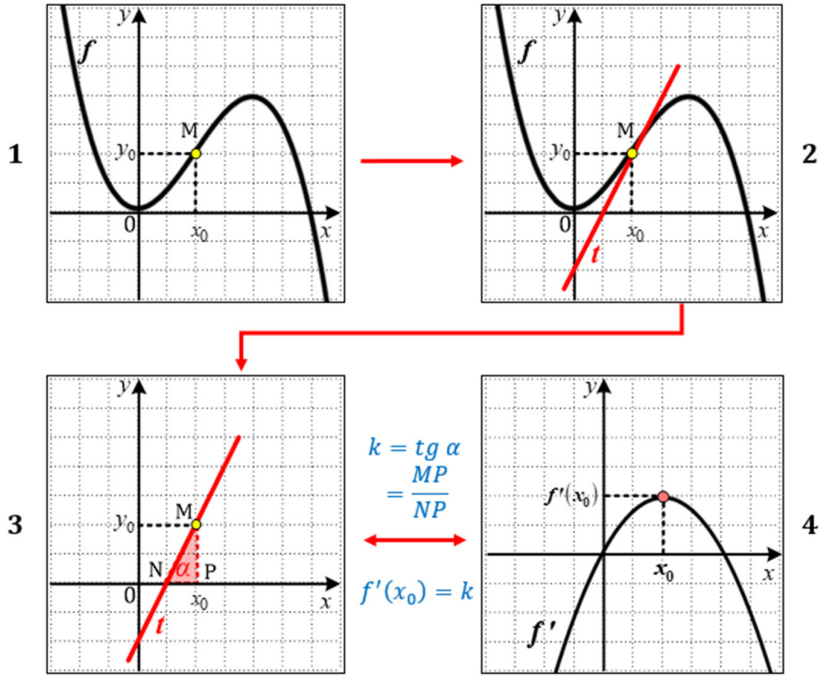


Figure 1. Graphical understanding of the derivative of a function

3.1. Typology of graphical tasks from the field of function derivatives

To address multiple representations and mathematical visualization in calculus, original tasks with graphical content were designed. Given their

broad methodological potential, tasks were classified by various criteria, forming a typology. Theoretical research identified four key characteristics of such tasks, determining their complexity and difficulty (Table 1).

Table 1. Factors influencing the complexity of requirements and the level of difficulty of tasks with graphical content and/or requirements

Lower Complexity / Lower Requirements	**Higher Complexity / Higher Requirements**
Working with the graph of a function	Working with the graph of derivative function
Considering a differentiable function	Considering a function that is not differentiable at all points in its domain
Considering local properties of the function/derivative function	Considering global properties of the function/derivative function
Interpreting graphical representations	Constructing graphical representations

The typology of tasks with graphical content is based on the first two criteria from Table 1. Four task types, labeled A, B, C, and D (fig. 2), are identified. Types A and B involve the graphical representation of a function, while C and D focus on its derivative function's graph. Tasks A and C use differentiable functions, whereas B and D involve non-differentiable functions. Given curriculum content and task complexity, tasks should be introduced sequentially: $A \rightarrow B \rightarrow C \rightarrow D$.

To vary task complexity within each type, two additional criteria are applied: consideration of the local/global properties of functions and derivatives, and interpretation/construction of graphical representations.

The task typology enables designing and selecting graphical tasks on function derivatives, creating a structured, progressively complex problem series.

There are series of graphical tasks including various contents from differential calculus: geometric meaning of the first derivative, monotonicity

of functions, local and global extrema, convexity. We will present tasks for the geometric meaning of the derivative – the concept of the tangent.

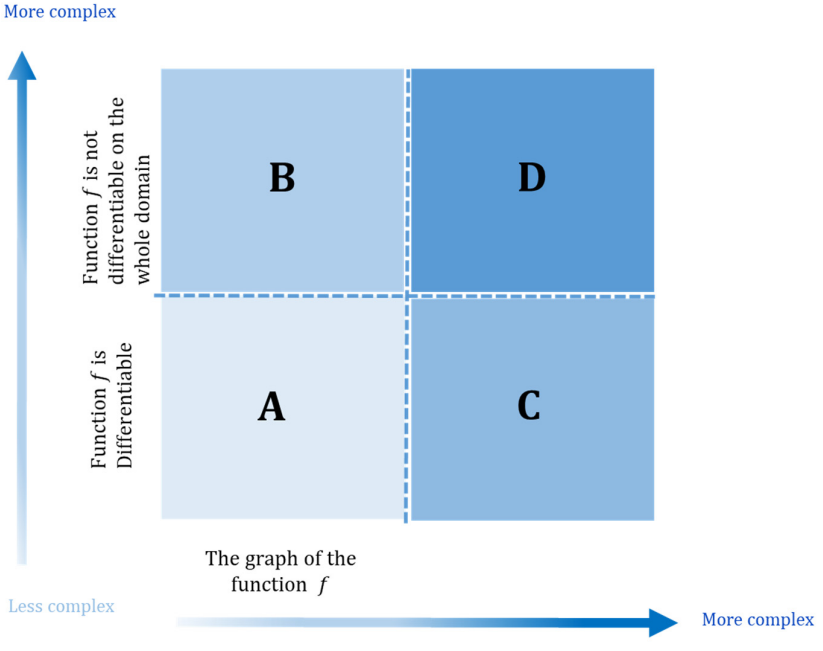


Figure 2. The typology of tasks with graphical content and/or requirements

3.2. The application of graphical tasks in teaching and learning derivatives of functions (the concept of the tangent)

The visual approach expands and enriches traditional teaching materials with a new type of tasks with graphical content and/or requirements. These tasks contain a visual component of the geometric interpretation of the first derivative, which is precisely what traditional tasks lack. For the treatment of the geometric interpretation of the first derivative, we have designed and created four types of tasks (Type A, B, C, and D) with graphical content and/or requirements. Below, we provided some examples of tasks of Type A, B, C, and D and explained in detail their characteristics.

Type A tasks

Type A tasks, noted in the literature (Asiala et al. 1997), provide a strong methodological basis for developing a graphical understanding of derivatives and testing students' knowledge of a function's derivative at a point.

These tasks help students interpret the slope of the tangent to the graph of a function as the value of the derivative function at a point. Since students typically understand slope algebraically, they may face difficulties with graphical tasks. Guidance is needed to ensure they extract relevant data from the graph by correctly reading the coordinate grid.

Type A tasks include a graphical representation of differentiable function (Gf) and its tangent (Gt). Based on this, students determine the derivative of the function at a point (Ad) and an algebraic local property (Af), such as function value, zeros, or local extrema (fig. 3).

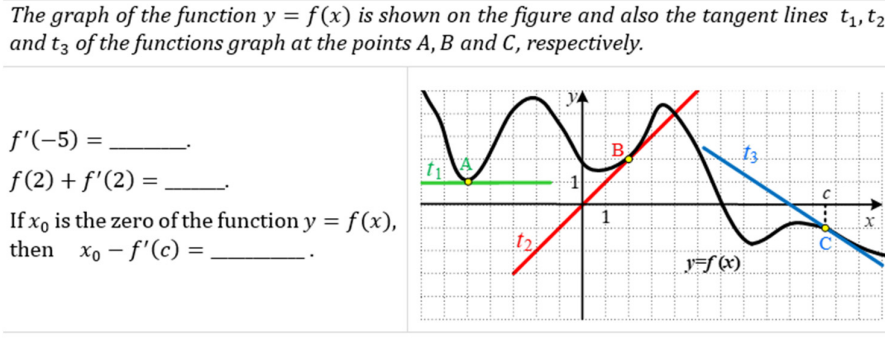


Figure 3. Type A task

The simplest Type A tasks are those where the angle between the tangent and the positive direction of the x −axis, and therefore the slope of the tangent can be directly determined from the graphical representation. In these tasks, the tangents to the graph are parallel to the x −axis (t_1), or form an angle of 45° with the positive direction of the x −axis (t_2), or the slope of the tangent can be calculated from the corresponding right-angled triangle (t_3), (fig. 4).

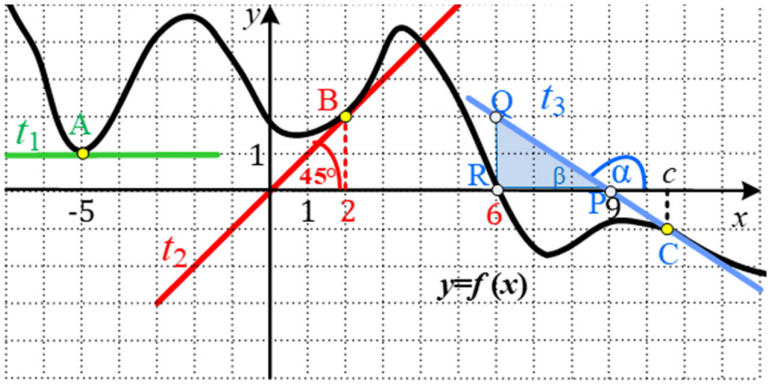


Figure 4. Type A task where the slope of the tangent can be directly determined

Type B tasks

Type B tasks involve a function's graph that is not differentiable across its entire domain, focusing on properties of its derivative. These properties are considered in both algebraic and graphical forms, and include domain, zeros, and sign.

In the problem-solving process, the students are required to draw or mentally visualize tangents on the graph, developing an internal representation of the tangent. In other words, these tasks require that students have an internal, mental representation of the tangent, which they can display as an external graphical representation through drawing, but also manipulate mentally without drawing (fig. 5).

In the process of solving the type B tasks, the following transitions are realized: algebraic property of the derivative function (Ad) → algebraic property of the tangent related to the slope coefficient (At) → geometric property of the tangent related to the position (angle) of the tangent relative to the x −axis (Gt) → construction of the corresponding tangents to the function's graph (Gt, Gf) → reading from the graphical representation and writing down the solution of the task (Ad).

The graph of the function $y = f(x)$ is shown on the figure.

1. At how many points does the graph of the derivative function f' intersect the x–axis?
2. Calculate the sum of all solutions to the equation $f'(x) = 0$.
3. How many integer values of the argument x are there such that the derivative function $y = f'(x)$ is positive?
4. Determine the sum of the integer solutions to the inequality $f'(x) < 0$.
5. Determine the domain of the function $y = f'(x)$.

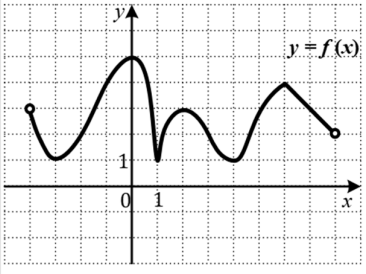


Figure 5. Type B task

When solving type B tasks, critical points of the function's graph are examined from the tangent's perspective, not as extrema or inflection points. Emphasis is placed on connecting the critical point, the (im)possibility of constructing a tangent, its slope, and the corresponding derivative property.

The solving process of type B tasks is represented by fig. 6. The first and second tasks are similar, differing only in representation – graphical for the first, algebraic for the second. Both require determining when the derivative of f equals zero based on its graph. Students first, from the condition $f'(x) = 0$, conclude about the slope of the tangent ($k = 0$), which they translate into the corresponding geometric property of the tangent: the tangent is parallel to the x-axis. Next, they need to construct all the horizontal tangents of the function's graph. For the given graph of the function, there are four points where the tangents are horizontal, which is also the solution to the first example. The solutions to the equation $f'(x) = 0$ are the first coordinates of these points, i.e., the solutions are $-3, 0, 2$, and 4. The sum of the solutions to the equation given in the second task is 3.

The first four tasks require examining some local property of the derivative function, while the fifth task considers a global property, the domain of the derivative function. It should be noted that determining the domain of the derivative function based on the graph of the function, or

"reading differentiability" from the graph, is not a formal proof but is very important for understanding this concept.

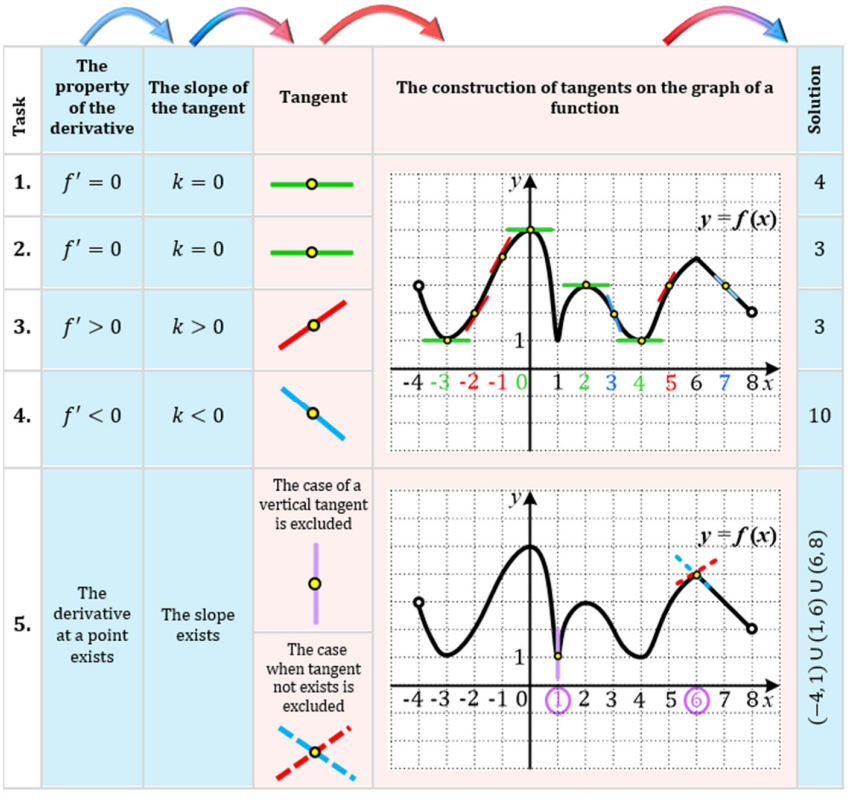


Figure 6. Type B task solving procedure

To solve the fifth task, knowledge of the graphical interpretation of the existence of a function's derivative at a point is needed. If at the point $(a, f(a))$ of the function's graph, a tangent that forms an angle $\alpha \neq 90°$ with the x –axis can be constructed, then the derivative function f' is defined at point a. In other words, there are two cases when the function does not have a derivative at a point: 1) when the tangent at the point on the function's graph is vertical, and 2) when the function's graph does not

have a tangent at that point. Based on the given graph, it can be observed that at point $(1,1)$, the tangent is vertical. At point $(6,4)$, there is no tangent on the graph because the left and right tangents are two different lines. At all other points on the graph, tangents exist and they are not vertical. Therefore, the derivative function is not defined for $x = 1$ and $x = 6$, and its domain is $(-4,1) \cup (1,6) \cup (6,8)$.

A complete graphical understanding of differential calculus requires reversible connections between concepts. Reversibility, a key aspect of mathematical thinking, supports relational understanding and flexibility. Type A and B tasks establish one-way connections from a function's graph to the graphical/algebraic properties of its derivative. Next, type C and D tasks reverse this process, analyzing function properties based on the derivative's function graph.

Type C tasks

In type C tasks, differentiable functions are considered. Based on the graph of the derivative function provided in the task, certain properties of the function f are examined, particularly those related to the tangent (fig. 7).

The function $y = f(x)$ is defined on the interval $(-6,7)$. The graph of its derivative function $y = f'(x)$ is shown on the figure.

1. At how many points are the tangents of the graph of the function $y = f(x)$ parallel to the x–axis?
2. Determine the sum of the x-coordinates of the points on the graph of the function $y = f(x)$ where the tangents are perpendicular to the line $x + 2y - 12 = 0$.
3. At how many points the tangents of the graph of the function $y = f(x)$ build an angle of 45° to the positive direction of x–axis?
4. Determine for which values of the argument x, the tangents of the graph of the function $y = f(x)$ build obtuse angles to the positive direction of x–axis.

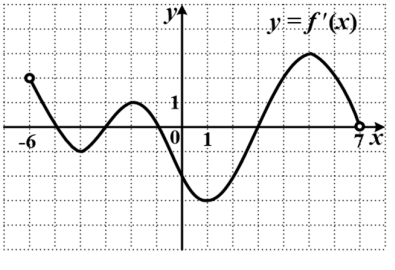


Figure 7. Type C task

The procedure for solving the given type C tasks is illustrated in fig. 8. In all four tasks, algebraic data about a specific property of the tangent (At) is provided.

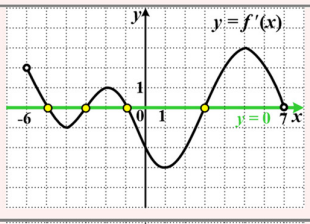
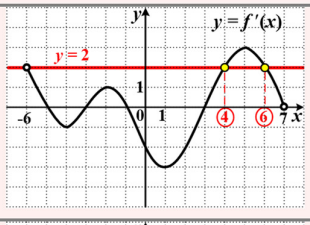
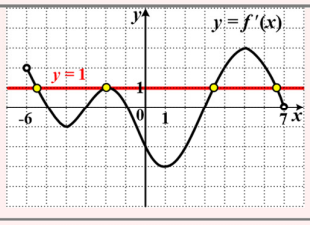
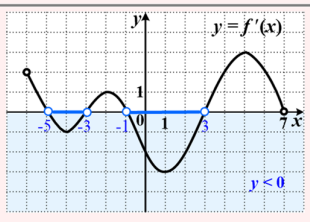
Task	The property of the tangent	The slope of the tangent	The property of the derivative function	Graphical representation of the properties of the derivative function	Solution
1.	Parallel to x −axis (the condition for the parallelism of lines)	$k = 0$	$f'(x) = 0$		4 points
2.	Perpendicular to the line: $x + 2y - 12 = 0$ (the condition for the perpendicularity of lines)	$k = 2$	$f'(x) = 2$		4 + 6 = 10
3.	Build an angle of $\alpha = 45°$ to x−axis $(k = tg\ \alpha)$	$k = 1$	$f'(x) = 1$		4 points
4.	Build an obtuse angle α to x−axis $(k = tg\ \alpha)$	$k < 0$	$f'(x) < 0$		$x \in (-5, -3) \cup (-1, 3)$

Figure 8. Type C task solving procedure

By applying knowledge of analytical geometry (conditions for parallelism/perpendicularity of lines, the relationship between the slope coefficient and the angle of inclination), the appropriate conclusion about the slope of the tangent is drawn, and consequently, about the algebraic property of the derivative function (Ad). This property must then be represented on the graph of the derivative function (Gd). In the final step, based on the graphical representation, the required set of solutions for the property of the derivative function (Ad), or for the property of the tangent considered in the given example (At), is read. The solving process involves transitions between concepts and/or representations in the order: $At \rightarrow Ad \rightarrow Gd \rightarrow Ad \rightarrow At$.

Type D tasks

In type D tasks, a function f that is not differentiable at all points in its domain is considered. The graph of the derivative function f' is given as input, and the task requirements focus on examining the properties of the function f, particularly those related to the tangent (fig. 9). Type D tasks require knowledge about the properties of the graph of the derivative function f' in the neighborhood of the points where the function f is not differentiable.

The function $y = f(x)$ is defined and continuous on the interval $(-6, 7)$. On the picture the graph of its derivative function $y = f'(x)$ is showed.

1. The tangent to the graph of the function $y = f(x)$ at the point $A\big(a, f(a)\big)$ is parallel to the y-axis. Determine the equation of this tangent.

2. The graph of the function $y = f(x)$ does not have a tangent at the point $B\big(b, f(b)\big)$. Determine b.

3. For which values of the argument x, do the tangents to the graph of the function $y = f(x)$ not form acute angles with the positive direction of the x-axis?

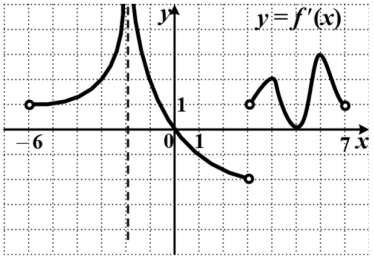


Figure 9. Type D task

Fig. 10 illustrates the procedures for solving the mentioned type D tasks. In the first task, it is necessary to determine the equation of the tangent to

the graph of the function f, which is parallel to the y −axis. If the tangent at the point $A(a, f(a))$ is vertical, then the left and right derivatives at the point $x = a$ approach infinity. This means that $x = a$ is both the left and right vertical asymptote of the graph of the derivative function f'. Since the line $x = -2$ is both the left and right vertical asymptote of the graph of the derivative function $y = f'(x)$, it follows that $a = -2$. The equation of the vertical tangent at the point $A(-2, f(-2))$ on the graph of the function f is $x = -2$.

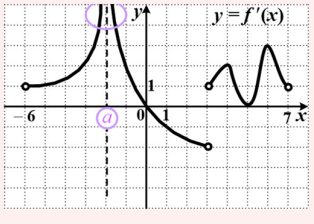
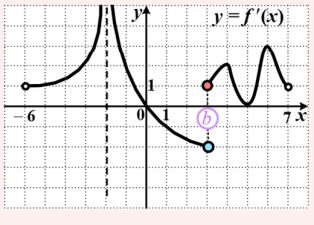
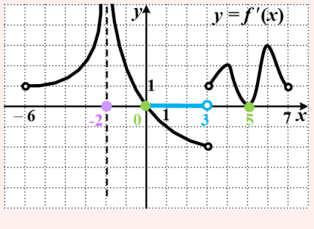
Task	Tangent	Properties of the derivative function	Graphical representation of the properties of the derivative function	Solution
1.	Vertical tangent at the point $A(a, f(a))$ 	$\lim_{x \to a-} f'(x) = \pm\infty$ $\lim_{x \to a+} f'(x) = \pm\infty$ $\Downarrow$ $x = a$ is left and right vertical asymptote to the derivative functions' f' graph		$x = -2$
2.	At the point $B(b, f(b))$ tangent does not exist 	The derivative function is not defined for $x = b$ where: $\lim_{x \to b-} f'(x) \neq \lim_{x \to b+} f'(x)$ and at least one of the limits is finite		$b = 3$
3.	  	$f'(x) = 0$ Case 1) $f'(x) < 0$		$x = -2 \vee x = 5 \vee 0 \leq x < 3$

Figure 10. Type D task solving procedure

In the second task, the case is considered where the graph of the function has no tangent at point B. This means that left and right tangents can be constructed at point B, but they are two different lines. In other words, the derivative function is not defined at the point $x = b$, where $\lim_{x \to b-} f'(x) \neq \lim_{x \to b+} f'(x)$, and at least one of these limits is finite.

In the third task, tangents to the graph of the function f are considered, which form angles that are not acute with the positive direction of the x-axis. This means that horizontal tangents, vertical tangents, and tangents with obtuse slope angles are examined. From the given geometric properties of the tangents, the corresponding algebraic properties of the derivative function are obtained. In the case of a horizontal tangent, $f'(x) = 0$ holds. If the tangent is vertical, then the limits $\lim_{x \to a-} f'(x)$ and $\lim_{x \to a+} f'(x)$ are infinite. If the slope angle of the tangent is obtuse, then $f'(x) < 0$. These algebraic properties of the derivative function should be translated into the corresponding properties of the graph of the derivative function (intersection of the graph with the x-axis, vertical asymptote, part of the graph below the x-axis), and based on the graphical representation, the required set of solutions can be determined.

4. Conclusions

Incorporating graphical tasks into calculus education enhances students' visual reasoning and conceptual understanding of mathematical concepts. The new typology of tasks introduced in this paper aims to address the existing imbalance in traditional teaching methods, which often prioritize algebraic and procedural approaches over graphical interpretations. By engaging students with visual representations, we can bridge the gap between symbolic manipulation and graphical understanding, fostering a more comprehensive and balanced mathematical education.

This approach not only improves students' ability to interpret and create graphical displays but also enhances their flexibility in transitioning between different representations of mathematical concepts. It encourages the development of higher-order thinking skills, such as problem-solving and critical analysis, which are essential for tackling novel and non-routine problems. Furthermore, integrating visual reasoning tasks into the

curriculum leverages the potential of digital technologies and educational software, making learning more interactive and engaging.

In summary, the proposed typology of graphical tasks provides a valuable framework for enhancing the teaching and learning of calculus, promoting a deeper and more holistic understanding of mathematics. By prioritizing visual reasoning alongside traditional methods, educators can create a more inclusive and effective learning environment that serves to diverse learning styles and needs.

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