https://doi.org/math2026-4-1-ptr

Science in Education
Научно-методически статии

A PROPERTY OF A TETRAHEDRON WITH A RIGHT TRIHEDRAL ANGLE

Dragomir Grozev, Nikolai Nikolov
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences

Abstract. A property of a tetrahedron with a right trihedral angle is considered. Several approaches to its proof are presented. For this purpose, auxiliary statements are discussed, which also have independent significance.

Keywords: stereometry; right trihedral angle; tetrahedron; Lagrange's formula; Euler's formula; de Gua's theorem; mathematical olympiad

1. Introduction

The idea for this article originates from Problem 11.3 of the Bulgarian Spring Mathematical Competition 2026, proposed by the second author. The problem is stereometric, which is something refreshing. Problems of this type have been deliberately neglected in recent decades because they are excluded from the content framework of the International Mathematical Olympiad (IMO). We hope that this will change after 2027, which would lead to the inclusion of new and interesting ideas.

Let us consider an acute-angled $\triangle ABC$. Let O and H be respectively the circumcenter of $\triangle ABC$ and the orthocenter of $\triangle ABC$. We want to find a point X in space such that the trihedral angle at the vertex X of $XABC$ is right. There are two such points, symmetric with respect to the plane determined by ABC. It turns out that for such a point X, the following relation holds

$$XH^2 + XO^2 = R^2, \tag{1}$$

where R is the radius of the circumcircle of $\triangle ABC$. This interesting fact is in fact the mentioned problem 11.3 from the Spring math competition 2026. Below we present four different approaches to proving (1). Before that, auxiliary statements are considered, which also have independent significance – Lagrange's formula, Euler's formula, and de Gua's theorem, etc.

The article is intended for students actively engaged in Olympiad mathematics, as well as for teachers and coaches who train them.

2. Some useful facts

Many of these results are standard in olympiad geometry; see, e.g. (Prasolov, 2001).

Proposition 1. *(Lagrange's formula) Let $A_1, A_2, \ldots, A_n$ be points in space, where the point A_i has mass $m_i, 1 \leq i \leq n$. Let G be the center of mass of these points. Then for an arbitrary point O in space we have*

$$\sum_{i=1}^{n} m_i OA_i^2 = \left(\sum_{i=1}^{n} m_i\right) OG^2 + \sum_{i=1}^{n} m_i GA_i^2. \tag{2}$$

Before proceeding to the proof, let us note that the above formula has applications in mechanics, and in geometry it is mainly used in the special case $m_i = 1, 1 \leq i \leq n$.

Proof. Since G is the center of mass of the points $A_1, \ldots A_n$, we have

$$\sum_{i=1}^{n} m_i \overrightarrow{GA_i} = 0. \tag{3}$$

From $\overrightarrow{OA_i} = \overrightarrow{OG} + \overrightarrow{GA_i}$, after squaring (dot product) we obtain

$$m_i OA_i^2 = m_i OG^2 + 2m_i \overrightarrow{OG} \cdot \overrightarrow{GA_i} + m_i GA_i^2.$$

Summing these equalities for $i = 1, 2, \ldots, n$ and using (3), the result follows. □

Let us consider two simple consequences of the above statement.

Corollary 1. *(median formula) For $\triangle ABC$ with midpoint M of BC we have*

$$2AM^2 = AB^2 + AC^2 - \frac{BC^2}{2}.$$

Proof. We apply Lagrange's formula for $O \equiv A$, points B, C and $m_1 = m_2 = 1$. Since M is the center of mass of the points B, C, we obtain

$$AB^2 + AC^2 = 2AM^2 + MB^2 + MC^2 = 2AM^2 + \frac{BC^2}{2}.$$

□

Corollary 2. *Let $A_1, A_2, \ldots, A_n$ be points in space, and let G be their center of mass. Then the set of points X for which*

$$\sum_{i=1}^{n} XA_i^2 = C,$$

where $C > 0$ is a constant, is either empty or a sphere with center at the point G.

Proof. Indeed, applying Lagrange's formula for $m_i = 1$, we obtain

$$C = \sum_{i=1}^{n} XA_i^2 = nXG^2 + \sum_{i=1}^{n} GA_i^2,$$

from which, $XG^2 = C - \sum_{i=1}^{n} GA_i^2$. Note that the expression on the right-hand side depends only on the points A_i, which shows that for $C \geq \sum_{i=1}^{n} GA_i^2$, the distance XG is constant. The converse also follows directly. □

Example 1. *Let ABC be a triangle with orthocenter H and circumcenter O, and let R be its circumradius. We seek to determine all points X in space such that*

$$XO^2 + XH^2 = R^2.$$

Let K denote the center of the nine-point (Feuerbach) circle, i.e., the midpoint of OH. For further properties of this circle, see (Coxeter & Greitzer, 1967).

As we established, the locus of points X satisfying a relation of the form $XO^2 + XH^2 = const$ is either empty or a sphere. Applying Lagrange's formula, we obtain

$$R^2 = XO^2 + XH^2 = 2XK^2 + KO^2 + KH^2 = 2XK^2 + \frac{1}{2}OH^2,$$

and hence

$$XK = \frac{1}{2}\sqrt{2R^2 - OH^2}.$$

Thus:

- *if $2R^2 - OH^2 > 0$, the locus of X is a sphere with center K and radius $\frac{1}{2}\sqrt{2R^2 - OH^2}$;*
- *if $2R^2 - OH^2 = 0$, the locus consists of the single point K;*
- *if $2R^2 - OH^2 < 0$, no such point X exists.*

A straightforward computation yields

$$2R^2 - OH^2 = R^2\big(1 + 8\cos A \cos B \cos C\big). \tag{4}$$

Hence, the nature of the locus is determined by the sign of the right-hand side of (4). In particular, if $\triangle ABC$ is acute, then $2R^2 - OH^2 > 0$, whereas for sufficiently obtuse triangles one has $2R^2 - OH^2 < 0$.

In the acute case, (4) *further implies*

$$XK = \frac{1}{2}\sqrt{2R^2 - OH^2} > \frac{R}{2}.$$

Therefore, the locus of X is a sphere centered at K that strictly contains the nine-point circle, whose radius is $R/2$, see (Coxeter & Greitzer, 1967).

Proposition 2. *(Euler's formula, see (Johnson, 2007)) For $\triangle ABC$ with orthocenter H and circumradius R we have*

$$OH^2 = 9R^2 - a^2 - b^2 - c^2 \tag{5}$$

Proof. Let G be the centroid of $\triangle ABC$. From Lagrange's formula (2), applied to the points O, A, B, C, we obtain

$$OA^2 + OB^2 + OC^2 = 3OG^2 + GA^2 + GB^2 + GC^2,$$

which gives

$$3R^2 = 3OG^2 + GA^2 + GB^2 + GC^2. \tag{6}$$

From $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = 0$, after squaring we have

$$GA^2 + GB^2 + GC^2 + 2(\overrightarrow{GA}\cdot\overrightarrow{GB} + \overrightarrow{GA}\cdot\overrightarrow{GC} + \overrightarrow{GB}\cdot\overrightarrow{GC}) = 0. \tag{7}$$

On the other hand,

$$\begin{aligned} a^2 &= (\overrightarrow{GC} - \overrightarrow{GB})^2 = GB^2 + GC^2 - 2\overrightarrow{GB}\cdot\overrightarrow{GC}, \\ b^2 &= (\overrightarrow{GA} - \overrightarrow{GC})^2 = GA^2 + GC^2 - 2\overrightarrow{GA}\cdot\overrightarrow{GC}, \\ c^2 &= (\overrightarrow{GB} - \overrightarrow{GA})^2 = GB^2 + GA^2 - 2\overrightarrow{GA}\cdot\overrightarrow{GB}. \end{aligned}$$

Summing the above equalities and using (7), we obtain

$$3(GA^2 + GB^2 + GC^2) = a^2 + b^2 + c^2.$$

Substituting into (6) we get

$$9OG^2 = 9R^2 - a^2 - b^2 - c^2.$$

Finally, since the points O, G, H lie on a line in this order (Euler line) and $OH = 3OG$, we obtain

$$OH^2 = 9R^2 - a^2 - b^2 - c^2.$$

□

Now we present a well-known formula for the length of the segment connecting a vertex of a triangle with its orthocenter.

Proposition 3. *Let ABC be an acute triangle with orthocenter H and circumradius R. Then*

$$AH^2 = 4R^2 - BC^2.$$

Proof. Let H' be the reflection of H with respect to AC. It is known that H' lies on the circumcircle. We have $AH = AH'$. Let $H'O$ intersect the circumcircle again at the point H''. Since $H'H''$ is a diameter, it is easy to see that $\angle H''H'A = \angle A$, i.e. $AH'' = BC$. From the Pythagorean theorem in $\triangle AH'H''$ we obtain

$$4R^2 = H'H''^2 = AH'^2 + AH''^2 = AH^2 + BC^2.$$

□

The following result can be viewed as a three-dimensional analogue of the Pythagorean theorem. To the best of the authors' knowledge, this orthocenter-based proof of de Gua's theorem does not appear in the literature

Proposition 4. *(de Gua's theorem) Let $ABCD$ be a tetrahedron with a right trihedral angle at the vertex D. Then for the faces we have*

$$S^2_{ABC} = S^2_{DAB} + S^2_{DBC} + S^2_{DAC}.$$

Proof. Let H be the orthocenter of $\triangle ABC$. From the theorem of the three perpendiculars it follows that D projects onto the plane of ABC at the point H. Denote by A_1, B_1, C_1 the feet of the perpendiculars from the vertices A, B, C to the sides of the triangle, respectively. Note that the triangles $\triangle ADA_1, BDB_1, CDC_1$ are right-angled at D, and DH is an altitude for each of them. From this it follows

$$DA_1^2 = A_1H \cdot A_1A\,;\ DB_1^2 = B_1H \cdot B_1B\,;\ DC_1^2 = C_1H \cdot C_1C.$$

Since $DA_1 \perp BC$, $DB_1 \perp AC$, $DC_1 \perp AB$, we obtain

$$4S^2_{DBC} = DA_1^2 \cdot BC^2 = A_1H \cdot BC \cdot AA_1 \cdot BC = 4S_{HBC} \cdot S_{ABC}.$$

Similarly, $4S^2_{DAC} = 4S_{HAC} \cdot S_{ABC}$ and $4S^2_{DAB} = 4S_{HAB} \cdot S_{ABC}$. Summing these equalities, we obtain

$$S^2_{DAB} + S^2_{DBC} + S^2_{DAC} = S_{ABC}(S_{HAB} + S_{HBC} + S_{HAC}) = S^2_{ABC}.$$

□

3. Applications

Let us now consider the following problem.

Problem 1. *(Spring competition 2026, Problem 11.3) Let $ABCD$ be a tetrahedron with a right trihedral angle at the vertex D. The point H is the orthocenter of $\triangle ABC$, and the circumcircle of $\triangle ABC$ has center O and radius R. Prove that $DO^2 + DH^2 = R^2$.*

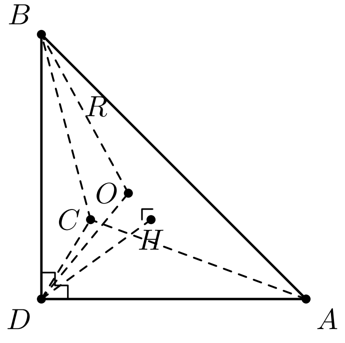

Figure 1: Problem 11.3.

We will prove the statement in 4 different ways. Let us begin with the most elegant one, in our opinion. It was found by two students during the competition.

First solution. Let H_1 be the foot of the perpendicular from A to BC, and let H_1' be the second intersection point of AH_1 with the circle k, see Fig. 1 and Fig. 2. We have

$$\begin{aligned} & DO^2 + DH^2 = R^2 \\ \iff\ & DH^2 + OH^2 + DH^2 = R^2 \\ \iff\ & 2DH^2 = R^2 - OH^2 = (R - OH)(R + OH) \end{aligned} \tag{8}$$

Note that the right-hand side equals the power of point H with respect to the circle k (Coxeter & Greitzer, 1967). This means $(R - OH)(R + OH) = AH \cdot HH_1'$. Moreover, it is known that $HH_1 = H_1H_1'$. Therefore

$$(R - OH)(R + OH) = AH \cdot HH_1' = 2AH \cdot HH_1.$$

From (8) we obtain

$$DO^2 + DH^2 = R^2 \iff DH^2 = AH \cdot HH_1.$$

Since $\triangle DAH_1$ is right-angled and DH is the altitude to AH_1, it follows that $DH^2 = AH \cdot HH_1$. Hence $DO^2 + DH^2 = R^2$.

Second solution. Let $a = BC$, $b = CA$, $c = AB$. It is easy to see that

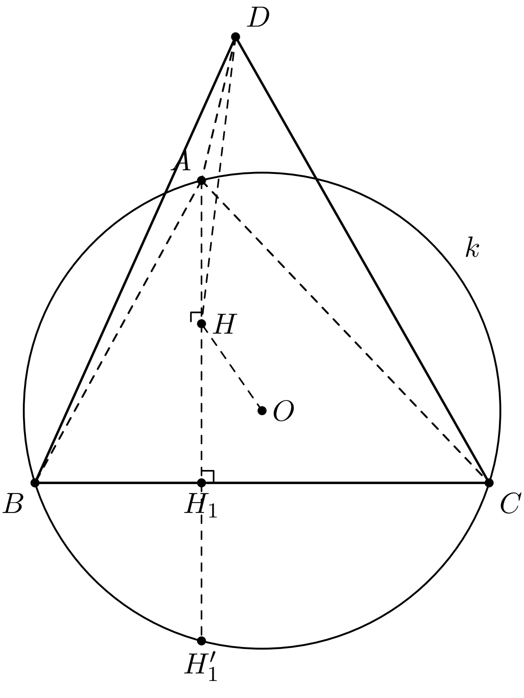

Figure 2: Configuration for the first solution.

$DH \perp (ABC)$ (e.g. by the theorem of the three perpendiculars). Then

$$DH^2 = DA^2 - AH^2 = DA^2 - 4R^2 + a^2 = DB^2 - 4R^2 + b^2 = DC^2 - 4R^2 + c^2 \tag{9}$$

and since $DB^2 + DC^2 = a^2$, $DC^2 + DA^2 = b^2$, $DA^2 + DB^2 = c^2$, we get

$$6DH^2 = 2(DA^2 + DB^2 + DC^2 + a^2 + b^2 + c^2 - 12R^2) = 3(a^2 + b^2 + c^2 - 8R^2).$$

From this and the Euler's formula, $OH^2 = 9R^2 - a^2 - b^2 - c^2$, it follows that

$$DO^2 + DH^2 = OH^2 + 2DH^2 = R^2.$$

□

Third solution. We introduce a coordinate system with origin at D and axes along the edges DA, DB, DC. Let $A = (a_1, 0, 0)$, $B = (0, a_2, 0)$, $C = (0, 0, a_3)$ and $O = (o_1, o_2, o_3)$. The plane (ABC) has equation $x_1/a_1 + x_2/a_2 + x_3/a_3 = 1$ and H is the projection of D onto it. Using the formula for the distance from a point to a plane, we obtain

$$DH = (a_1^{-2} + a_2^{-2} + a_3^{-2})^{-1/2}. \tag{10}$$

Further, since $O \in (ABC)$, we have

$$\frac{o_1}{a_1} + \frac{o_2}{a_2} + \frac{o_3}{a_3} = 1. \tag{11}$$

Moreover,

$$(o_1 - a_1)^2 + o_2^2 + o_3^2 = o_1^2 + (o_2 - a_2)^2 + o_3^2 = o_1^2 + o_2^2 + (o_3 - a_3)^2 = R^2.$$

Since, $DO^2 = o_1^2 + o_2^2 + o_3^2$, we obtain

$$a_i^2 - 2o_i a_i = R^2 - DO^2, \text{ i.e. } 1 - 2\frac{o_i}{a_i} = \frac{R^2 - DO^2}{a_i^2}, \quad i = 1, 2, 3.$$

It remains to sum these three equalities and use (10) and (11).

Fourth solution. Let $x = DA$, $y = DB$, $z = DC$ and $S = S_{ABC}$. The idea is to express DH^2, R^2 and DO^2 in terms of x, y, z. From the theorem of the three perpendiculars it follows that $DH \perp (ABC)$. Writing the volume of $ABCD$ in two ways, we obtain

$$DH = \frac{xyz}{2S}. \tag{12}$$

From $R = \frac{a.b.c}{4S}$ we have

$$R^2 = \frac{(x^2 + y^2)(y^2 + z^2)(x^2 + z^2)}{16S^2}. \tag{13}$$

Let G denote the centroid of $\triangle ABC$. We have

$$\overrightarrow{DG} = \frac{1}{3}\left(\overrightarrow{DA} + \overrightarrow{DB} + \overrightarrow{DC}\right),$$

from which

$$DG^2 = \frac{1}{9}(x^2 + y^2 + z^2).$$

$$GH^2 = DG^2 - DH^2 = \frac{1}{9}(x^2 + y^2 + z^2) - \frac{x^2y^2z^2}{4S^2},$$

Since O, G and H lie on the Euler line and $OG : GH = 1 : 2$, we have $HO = \frac{3}{2}GH$ and

$$HO^2 = \frac{9}{4}HG^2 = \frac{1}{4}(x^2 + y^2 + z^2) - \frac{9}{16}\frac{x^2y^2z^2}{S^2}.$$

From $DO^2 = DH^2 + HO^2$ and (12) we have

$$DO^2 = \frac{x^2y^2z^2}{4S^2} + \frac{1}{4}(x^2+y^2+z^2) - \frac{9}{16}\frac{x^2y^2z^2}{S^2},$$

$$DO^2 = \frac{1}{4}(x^2+y^2+z^2) - \frac{5}{16}\frac{x^2y^2z^2}{S^2}.$$

$$DO^2+DH^2 = \frac{1}{4}(x^2+y^2+z^2) - \frac{5}{16}\frac{x^2y^2z^2}{S^2} + \frac{x^2y^2z^2}{4S^2} = \frac{1}{4}(x^2+y^2+z^2) - \frac{1}{16}\frac{x^2y^2z^2}{S^2}.$$

From de Gua's theorem,

$$S^2 = S^2_{DAB} + S^2_{DBC} + S^2_{DAC},$$

$$S^2 = \frac{1}{4}\left(x^2y^2 + x^2z^2 + y^2z^2\right),$$

from which

$$DO^2 + DH^2 = \frac{1}{4}(x^2+y^2+z^2) - \frac{1}{4}\frac{x^2y^2z^2}{(x^2y^2+x^2z^2+y^2z^2)}. \tag{14}$$

From (13) we have

$$R^2 = \frac{(x^2+y^2)(y^2+z^2)(x^2+z^2)}{4\left(x^2y^2+x^2z^2+y^2z^2\right)}. \tag{15}$$

After carrying out the computations in (14) and (15), we obtain

$$DO^2 + DH^2 = R^2.$$

Acknowledgements

The second named author was partially supported by the Bulgarian National Science Fund, Ministry of Education and Science of Bulgaria under contract KP-06-N82/6.

REFERENCES

Coxeter, H. S. M., Greitzer, S. L. (1967). *Geometry Revisited.* MAA.
Johnson, R. A. (2007). *Advanced Euclidean Geometry.* Dover.
Prasolov, V. V. (2001). *Problems in Geometry.* AMS.

✉ **Dragomir Grozev**
ORCID iD: 0009-0004-6365-2388
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Sofia, Bulgaria
E-mail: drago.grozev@gmail.com

✉ **Nikolai Nikolov**
ORCID iD: 0000-0002-4168-7625
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Sofia, Bulgaria
Faculty of Information Sciences
State University of Library Studies and Information
Sofia, Bulgaria
E-mail: nik@math.bas.bg