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A METHOD FOR SOLVING EXTREMAL PROBLEMS IN ALGEBRA AND STEREOMETRY

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Abstract. In this article, we propose a method for solving extremal problems in algebra and stereometry. The method is applied to problems that can be addressed using basic elementary inequalities and trigonometric equalities in triangles. These problems are reduced to finding the minimum or maximum value of a function. The aim of the method is to construct a systematic algorithm for solving extremal problems.

Keywords: extremal problems; smallest value of a function; largest value of a function

1. Introduction

The extremal problems occupy a central place in modern mathematics and its applications, especially in the fields of combinatorics, optimization, graph theory and analysis. They arise naturally when investigating the maximum or minimum values of a given quantity under certain constraints. From the classical problems of the calculus of variations to contemporary issues in algorithms and network theory, extremal problems not only provide rich opportunities for theoretical research, but also find widespread application in science, engineering, economics and computer science. They occupy a special place in the training and development of the mathematical thinking. They require not only the mastery of basic concepts and techniques, but also analytic ability, strategic planning and a creative approach to problem solving.

Extremal problems in algebra and stereometry are an important part of mathematical education and research practice. They are most often associated with finding the largest or smallest values of given expressions, proving inequalities and optimizing geometric configurations in the space. These types of tasks not only develop logical and analytical thinking, but also play an essential role in preparing for mathematical competitions and in the formation of key mathematical intuitions. They play an important role in mathematics education, in the ability to model mathematically and abstract thinking.

In algebra, extremal problems often involve the study of inequalities, expressing variables through restrictions and using techniques such as symmetrization, substitution, or applying known inequalities (such as those of Cauchy–Buniakovsky, arithmetic mean, geometric mean, etc.). In stereometry, extremal problems are associated with optimizing volumes, surfaces or distances between objects in the space, often requiring spatial imagination, geometric intuition and the use of coordinate methods.

The present article aims to apply a method presented in (Dimitrov, Kodzha 2025) to solve extremal problems in algebra and stereometry through selected examples.

2. Problems

In the article we consider extremal problems in algebra and stereometry, for the solution of which we use the ideas in (Dimitrov, Kodzha 2025).

The first two problems are extremal problems in algebra.

Problem 1. (Kolev, Georgiev, Kokinova 2020) If x, y, and z are non-negative numbers and $x + y + z = 12$, then find the largest value of the expression $\sqrt{2x} + \sqrt{4y} + \sqrt{6z}$.

In (Kolev, Georgiev, Kokinova 2020) the following solution to the problem is proposed, which uses an inequality between arithmetic mean and geometric mean:

Solution. (Kolev, Georgiev, Kokinova 2020) We apply the inequality $\mathbb{G} \leq \mathbb{A}$ between arithmetic mean and geometric mean:

$$\sqrt{2x} \leq \frac{2+x}{2}; \quad \sqrt{4y} \leq \frac{4+y}{2}; \quad \sqrt{6z} \leq \frac{6+z}{2}.$$

Equalities are reached at $x = 2$, $y = 4$, and $z = 6$. Then

$$\sqrt{2x}+\sqrt{4y}+\sqrt{6z} \leq \frac{2+x}{2}+\frac{4+y}{2}+\frac{6+z}{2} = \frac{x+y+z+12}{2} = \frac{12+12}{2} = 12.$$

The largest value of the expression $\sqrt{2x} + \sqrt{4y} + \sqrt{6z}$ is 12 at $x = 2$, $y = 4$, and $z = 6$.

Now we will propose a solution to the problem using the method in (Dimitrov, Kodzha 2025).

Solution. From $x + y + z = 12$, we get $z = 12 - x - y$ and

$$\sqrt{2x} + \sqrt{4y} + \sqrt{6z} = \sqrt{2x} + \sqrt{4y} + \sqrt{72 - 6x - 6y}.$$

Let

$$f = \sqrt{2x} + \sqrt{4y} + \sqrt{72 - 6x - 6y}. \tag{1}$$

We will examine (1), first as a function of x, and then as a function of y. Let

$$f(x) = \sqrt{2x} + \sqrt{4y} + \sqrt{72 - 6x - 6y}; \quad 0 \leq x \leq 12, \tag{2}$$

where $0 \leq y \leq 12$ is a parameter. The derivative of (2) is

$$f^{'}(x) = \frac{\sqrt{24 - 2x - 2y} - \sqrt{6x}}{2\sqrt{x(12 - x - y)}}$$

and the function has one extremal point

$$x = \frac{12 - y}{4}, \tag{3}$$

in which there is a maximum. Thus we obtain that function (2) has its largest value at point (3).

Substituting the value of x from (3) into (1), we explore a new function of y

$$f(y) = 2(\sqrt{y} + \sqrt{24 - 2y}); \quad 0 \leq y \leq 12. \tag{4}$$

The derivative of (4) is

$$f^{'}(y) = \frac{\sqrt{12 - y} - \sqrt{2y}}{\sqrt{y(12 - y)}}$$

and has one extremal point

$$y = 4, \tag{5}$$

in which there is a maximum. Now we substitute the value of y from (5) into (3) and we obtain the value of x at the maximum point $x = 2$.

Thus we found the largest value of (1) in both variables. Finally, we can conclude that the largest value of the expression $\sqrt{2x}+\sqrt{4y}+\sqrt{6z}$ is reached at $x = 2$, $y = 4$, and $z = 12 - x - y = 12 - 2 - 4 = 6$, and its equal to 12.

Problem 2. (Kolev, Georgiev, Kokinova 2020) If for the non-negative numbers a, b, and c the equality $a + b + c = 6$ is true, then find the smallest value of the expression $a^4 + b^4 + c^4$.

In (Kolev, Georgiev, Kokinova 2020) the following solution to the problem is proposed, which uses an inequality between arithmetic mean and root mean square:

Solution. (Kolev, Georgiev, Kokinova 2020) From the inequality between root mean square and arithmetic mean
$\mathbb{K}_3 \geq \mathbb{A}_3$, we get $\sqrt{\dfrac{a^2+b^2+c^2}{3}} \geq \dfrac{a+b+c}{3} = \dfrac{6}{3} = 2$. Therefore $\dfrac{a^2+b^2+c^2}{3} \geq 4$ and $a^2 + b^2 + c^2 \geq 12$. Equality is achieved at $a = b = c = 2$. Thus $\left(a^2 + b^2 + c^2\right)_{min} = 12$.
We apply the inequality $\mathbb{K}_3 \geq \mathbb{A}_3$ for numbers a^2, b^2 and c^2, we get
$\sqrt{\dfrac{a^4+b^4+c^4}{3}} \geq \dfrac{a^2+b^2+c^2}{3}$. From $a^2 + b^2 + c^2 \geq 12$,
it follows that $\sqrt{\dfrac{a^4+b^4+c^4}{3}} \geq \dfrac{a^2+b^2+c^2}{3} \geq 4 \Leftrightarrow a^4+b^4+c^4 \geq 48$. Equality is achieved at $a = b = c = 2$. Finally $\left(a^4 + b^4 + c^4\right)_{min} = 48$.

Our solution to the problem using the method in (Dimitrov, Kodzha 2025) looks like this:

Solution. From $a + b + c = 6$, we get $c = 6 - a - b$. Let

$$f = a^4 + b^4 + c^4 = a^4 + b^4 + (6 - a - b)^4; \quad 0 \leq a \leq 6, \;\; 0 \leq b \leq 6. \qquad (6)$$

We first examine (6) as a function of a. The derivative of (6) can be written in the form

$$f^{'}(a) = 4(2a + b - 6)(a^2 + (b - 6)a + b^2 - 12b + 36)$$

and the function has a minimum for

$$a = \frac{6-b}{2}. \tag{7}$$

We substitute the value of a from (7) into (6) and obtain a function of b

$$f(b) = b^4 + 2\left(\frac{6-b}{2}\right)^4. \tag{8}$$

The derivative of (8) reduces to the form

$$f^{'}(b) = \frac{9(b-2)(b^2+12)}{2}$$

and the function has a minimum for

$$b = 2. \tag{9}$$

Substituting (9) into (7), we obtain the value of a at the minimum point $a = 2$. Thus, we have established that the function has the smallest value at

$$a = 2 \text{ and } b = 2. \tag{10}$$

Hence $c = 2$.

We obtained that the smallest value is reached at $a = b = c = 2$. Substituting (10) into (6), we find the smallest value $(a^4 + b^4 + c^4)_{min} = 48$.

The next two problems we will consider are extremal problems in stereometry.

Problem 3. (Kolarov, Lesov 2013, Kolarov, Lesov 2013) Find the smallest surface area of a rectangular parallelepiped with volume V.

Solution. From $V = abc$, we get $c = \dfrac{V}{ab}$. Let

$$S = 2ab + 2ac + 2bc = 2ab + \frac{2V}{b} + \frac{2V}{a}; \quad a > 0, \quad b > 0. \tag{11}$$

Let's first examine (11) as a function of a. We calculate the first derivative and get

$$S^{'}(a) = \frac{2a^2b - 2V}{a^2}.$$

The function has a minimum for

$$a = \sqrt{\frac{V}{b}}. \tag{12}$$

We substitute (12) into (10) and examine the new function

$$S(b) = 4\sqrt{Vb} + \frac{2V}{b} \tag{13}$$

for a minimum. The derivative of (13) is written in the form

$$S^{'}(b) = \frac{2V(b^2 - \sqrt{Vb})}{b^2\sqrt{Vb}}$$

and has a minimum for

$$b = \sqrt[3]{V}. \tag{14}$$

We substitute (14) into (12) and get $a = \sqrt[3]{V}$.

The smallest value of (11) is reached at $a = b = c = \sqrt[3]{V}$. Finally, of all rectangular parallelepipeds with a given volume V, the cube has the smallest surface area.

Problem 4. Find the rectangular parallelepiped of largest volume inscribed in a sphere with radius R (Figure 1).

Solution. Let the inscribed parallelepiped be $ABCDA_1B_1C_1D_1$, with dimensions $AB = a$, $BC = b$ and $CC_1 = c$. In $\triangle ABC$, by the Pythagorean theorem, we get $AC = \sqrt{a^2 + b^2}$. Since $AC_1 = 2R$ and the Pythagorean theorem in $\triangle ACC_1$, it follows that $a^2 + b^2 + c^2 = 4R^2$. Hence $c^2 = 4R^2 - a^2 - b^2$, that is $c = \sqrt{4R^2 - a^2 - b^2}$.

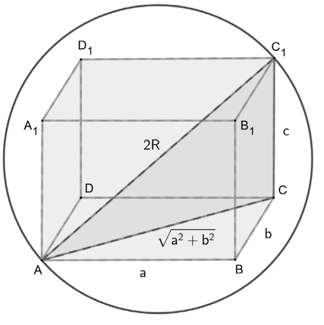

Figure 1. Rectangular parallelepiped inscribed in a sphere

For the volume of the parallelepiped we have

$$V = abc = ab\sqrt{4R^2 - a^2 - b^2}; \quad 0 < a < 2R,\ 0 < b < 2R. \tag{15}$$

We consider (15) first as a function of a. Its derivative is equal to

$$V^{'}(a) = b.\frac{4R^2 - 2a^2 - b^2}{\sqrt{4R^2 - a^2 - b^2}}$$

and reaches a maximum for

$$a = \sqrt{\frac{4R^2 - b^2}{2}}. \tag{16}$$

We substitute (16) into (15) and examine the new function, which is written in the form

$$V(b) = \frac{b(4R^2 - b^2)}{2}. \tag{17}$$

The derivative of (17) is

$$V^{'}(b) = \frac{4R^2 - 3b^2}{2} \tag{18}$$

and reaches a maximum for

$$b = \frac{2R}{\sqrt{3}}. \tag{19}$$

Finally, we substitute (19) into (16) and get $a = \dfrac{2R}{\sqrt{3}}$. Also $c = \dfrac{2R}{\sqrt{3}}$.

Thus, we found that (15) has the largest value for $a = b = c = \dfrac{2R}{\sqrt{3}}$. Finally, of all the rectangular parallelepipeds inscribed in a sphere the cube has the largest volume.

3. Conclusions

The study of extremal problems in algebra and stereometry reveals not only their theoretical depth, but also their broad potential for the development of logical and creative thinking. Through the analysis of specific examples, it was shown that these problems combine classical mathematical approaches with innovative methods of proof and optimization. They have important applications both in mathematics education and in preparation for competitions, where the ability to find limit values or optimal configurations plays a key role.

Extremal problems motivate students to build on their knowledge, develop analytical strategies, and build a sense of the structure and relationships in mathematical objects. The present study highlights the need for systematic inclusion of such tasks in the learning process and for further theoretical exploration of their potential in different mathematical contexts.

The study of extremal problems in algebra and stereometry reveals the richness and diversity of mathematical strategies through which maximum or minimum values are sought under given conditions. These tasks require not only solid knowledge, but also the ability to think abstractly, spatial imagination, and a non-standard approach to problem solving.

In algebra, extremal problems are often related to inequalities, symmetries, parametric dependencies, and optimizations in real sets. On the other hand, stereometric extremal problems emphasize the geometric relationships between volume, surface area, angles, and the location of objects in space. They reveal the deep connection between algebraic expressions and geometric constructions, which makes this type of task particularly suitable for developing interdisciplinary connections and integrated thinking.

Extremal problems not only test students' knowledge and skills, but also encourage them to apply a wider range of methods – from classical algebra and geometry to elements of mathematical analysis and logic. They serve as a bridge between standard curriculum content and competitive mathematics, encouraging independent inquiry and exploration.

The extremal problems can establish themselves as an important element not only in the education of talented students, but also in building a deeper mathematical culture among all learners.

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